

Quantum mechanics in general quantum systems (II): perturbation theory

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We propose an improved scheme of perturbation theory based on our exact solution [An Min Wang, quant-ph/0602055] in general quantum systems independent of time. Our elementary start-point is to introduce the perturbing parameter as late as possible. Our main skills are Hamiltonian redivision so as to overcome a flaw of the usual perturbation theory, and the perturbing Hamiltonian matrix product decomposition in order to separate the contraction and anti-contraction terms. Our calculational technology is the limit process for eliminating apparent divergences. Our central idea is “dynamical rearrangement and summation” for the sake of the partial contributions from the high order even all order approximations absorbed in our perturbed solution. Consequently, we obtain the improved forms of the zeroth, first, second and third order perturbed solutions absorbing the partial contributions from the high order even all order approximations of perturbation. Then we deduce the improved transition probability. In special, we propose the revised Fermi’s golden rule. Moreover, we apply our scheme to obtain the improved forms of perturbed energy and perturbed state. In addition, we study an easy understanding example of two-state system to illustrate our scheme and show its advantages. All of this implies the physical reasons and evidences why our improved scheme of perturbation theory are actually calculable, operationally efficient, conclusively more accurate. Our improved scheme is the further development and interesting application of our exact solution, and it has been successfully used to study on open system dynamics [An Min Wang, quant-ph/0601051].

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I. INTRODUCTION

The known perturbation theory [1, 2] is an extremely important tool for describing real quantum systems, as it turns out to be very difficult to find exact solutions to the Schrödinger equation for Hamiltonians of even moderate complexity. Recently, we see the dawn to overcome this difficulty because we obtained the exact solution of the Schrödinger equation [1] in general quantum systems independent of times [3]. However, this does not mean that the perturbation theory is unnecessary because our exact solution is still an infinite power series of perturbation. Our solution is called “exact one” in the sense including all order approximations of perturbation. In practice, if we do not intend to apply our exact solution to investigations of the formal theory of quantum mechanics, we often need to cut off our exact solution series to some given order approximation in the calculations of concrete problems. Perhaps, one argues that our exact solution will back to the usual perturbation theory, and it is, at most, an explicit form that can bring out the efficiency amelioration. Nevertheless, the case is not so. Such a view, in fact, ignores the significance of the general term in an infinite series, and forgets the technologies to deal with an infinite series in the present mathematics and physics. From our point of view, since the general term is known, we can systematically and reasonably absorb the partial contributions from some high order even all order approximations to the lower order approximations just like one has done in quantum field theory via summation over a series of different order but similar feature Feynman figures. In this paper, based on such a method we develop our “dynamical arrangement and summation” idea, and then propose an improved scheme of perturbation theory via introducing several useful skills and methods.

It is very interesting that we find a flaw in the usual perturbation theory, that is, the perturbing parameter is introduced too early so that the contributions from the high order even all order approximations of the diagonal and off-diagonal elements of the perturbing Hamiltonian matrix are, respectively, inappropriately dropped and prematurely cut off. For some systems, the influences on the calculation precision from this flaw can be not neglectable with the evolution time increasing. This motivates us to set our start-point to introduce the perturbing parameter as late as possible in order to guarantee the generality and precision. It is natural from a mathematics view if we think the

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perturbing parameter in a general perturbation theory is a formal multiplier. Based on this start point, we propose Hamiltonian redivision skill and further methods so as to overcome the above flaw in the usual perturbation theory, viz. the Hamiltonian redivision makes the contributions from all order approximations of the diagonal elements of the perturbing Hamiltonian matrix can be absorbed in our improved form of perturbed solution. Hence, this skill advances the calculation precision in theory, extends the application range for the perturbation theory and can remove degeneracies in some systems.

Since our exact solution series has apparent divergences, we provide the methods of perturbing Hamiltonian matrix product decomposition in order to separate the contraction terms with apparent divergences and anti-contraction terms without apparent divergences. Here, “apparent” refers to an untured thing, that is, the apparent divergences are not real singularities and they can be eliminated by mathematical and/or physical methods, while the “perturbing Hamiltonian matrix” refers to the representation matrix of the perturbing Hamiltonian in the unperturbed Hamiltonian representation. Then, by the limit process we can eliminate these apparent divergences in the contraction terms. Furthermore we apply “dynamical rearrangement and summation” idea for the sake of the partial contributions from the high order even all order approximations absorbed in our perturbed solution. In terms of these useful ideas, skills and methods we build an improved scheme of perturbation theory. Without any doubt, they are given definitely dependent on our exact solution. In fact, our exact solution inherits the distinguished feature in a c -number function form just like the Feynman [4] path integral expression and keeps the advantage in Dyson series [5] that is a power series of perturbation. At the same time, our exact solution is so explicit that when applying it to a concrete quantum system, all we need to do is only the calculations of perturbing Hamiltonian matrix and the limitations of primary functions.

As well known, a key idea of the existed perturbation theory to research the time evolution of system is to split the Hamiltonian of system into two parts, that is

$$H = H_0 + H_1, \quad (1)$$

where the eigenvalue problem of so called unperturbed Hamiltonian H_0 is solvable, and so-call perturbing Hamiltonian H_1 is the rest part of the Hamiltonian. In other words, this splitting is chosen in such a manner that the solutions of H_0 are known as

$$H_0|\Phi^\gamma\rangle = E_\gamma|\Phi^\gamma\rangle, \quad (2)$$

where $|\Phi^\gamma\rangle$ is the eigenvector of H_0 and E_γ is the corresponding eigenvalue. Whole $|\Phi^\gamma\rangle$, in which γ takes over all possible values, form a representation of the unperturbed Hamiltonian. It must be emphasized that the principle of Hamiltonian split is not just the best solvability mentioned above in more general cases. If there are degeneracies, the Hamiltonian split is also restricted by the condition that the degeneracies can be completely removed via the usual diagonalization procedure of the degenerate subspaces and the Hamiltonian redivision proposed in this paper, or specially, if the remained degeneracies are allowed, it requires that the off-diagonal element of the perturbing Hamiltonian matrix between any two degenerate levels are always vanishing so as to let our improved scheme of perturbation theory work well. As an example, it has been discussed in our serial study [6]. In addition, if the cut-off approximation of perturbation is necessary, it requires that for our improved scheme of perturbation theory, the off-diagonal elements of H_1 matrix is small enough compared with the diagonal element of $H = H_0 + H_1$ matrix in the unperturbed representation.

Nevertheless, there are some known shortcomings in the existed perturbation theory, for example, when H_1 is not so small compared with H_0 that the high order approximations should be considered, and/or when the partial contributions from the higher order approximations become relatively important to the studied problems, and/or the evolution time is long enough, the usual perturbation theory might be difficult to calculate to an enough precision in an effective way, even not feasible practically since the lower approximation might break the physical symmetries and/or constraints. In order to overcome these difficulties and problems, we recently study and obtain the exact solution in general quantum systems via explicitly expressing the time evolution operator as a c -number function and a power series of perturbation including all order approximations [3]. In this paper, our purpose is to build an improved scheme of perturbation theory based on our exact solution [3] so that the physical problems are more accurately and effectively calculated. For simplicity, we focus on the cases of Schrödinger dynamics [1]. It is direct to extend our improved scheme to the cases of the von Neumann dynamics [7].

Just well-known, quantum dynamics and its perturbation theory have been sufficiently studied and have many successful applications. Many famous physicists created their nice formulism and obtained some marvelous results. An attempt to improve its part content or increase some new content as well as some new methods must be very difficult in their realizations. However, our endeavors have obtained their returns, for examples, our exact solution [3], perturbation theory and open system dynamics [6] in general quantum systems independent of time.

In this paper, we start from proposing our ideas, skills and methods. We expressly obtain the improved forms of the zeroth, first, second and third order approximations of perturbed solution absorbing partial contributions from the

high order even all order approximations, finding the improved transition probability, specially, the revised Fermi's golden rule, and providing an operational scheme to calculate the perturbed energy and perturbed state. Furthermore, by studying a concrete example of two state system, we illustrate clearly that our solution is more efficient and more accurate than the usual perturbative method. In short, our exact solution and perturbative scheme are formally explicit, actually calculable, operationally efficient, conclusively more accurate (to the needed precision).

This paper is organized as the following: in Sec. II we find a flaw of the usual perturbation theory and introduce Hamiltonian redivision to overcome it. Then, we propose the skill of the perturbing Hamiltonian matrix product decomposition in order to separate the contraction terms with apparent divergences and anti-contraction terms without apparent divergences. By the limit process we can eliminate these apparent divergences. More importantly, we use the “dynamical rearrangement and summation” idea so that the partial contributions from the high order even all order approximations are absorbed in our perturbed solution and the above flaw is further overcome; in Sec. III we obtain the improved forms of the zeroth, first, second and third order perturbed solutions of dynamics absorbing partial contributions from the high order even all order approximations; in Sec. IV we deduce the improved transition probability, specially, the revised Fermi's golden rule. In Sec. V we provide a scheme to calculate the perturbed energy and the perturbed state; in Sec. VI we study an example of two state system in order to concretely illustrate our solution to be more effective and more accurate than the usual method; in Sec. VII we summarize our conclusions and give some discussions. Finally, we write an appendix as well as a supplementary where some expressions are calculated in order to derive out the improved forms of perturbed solutions.

II. SKILLS AND METHODS IN THE IMPROVED SCHEME OF PERTURBATION THEORY

In our recent work [3], by splitting a Hamiltonian into two parts, using the solvability of eigenvalue problem of one part of the Hamiltonian, proving an useful identity and deducing an expansion formula of operator binomials power, we obtain an explicit and general form of the time evolution operator in the representation of solvable part (unperturbed part) of the Hamiltonian. Then we find out an exact solution of the Schrödinger equation in general quantum systems independent of time

$$|\Psi(t)\rangle = \sum_{l=0}^{\infty} \mathcal{A}_l(t) |\Psi(0)\rangle = \sum_{l=0}^{\infty} \sum_{\gamma, \gamma'} A_l^{\gamma\gamma'}(t) \left[\langle \Phi^{\gamma'} | \Psi(0) \rangle \right] |\Phi^{\gamma}\rangle, \quad (3)$$

where

$$\mathcal{A}_l(t) = \sum_{\gamma, \gamma'} A_l^{\gamma\gamma'}(t) |\Phi^{\gamma}\rangle \langle \Phi^{\gamma'}|, \quad (4)$$

$$A_0^{\gamma\gamma'}(t) = \sum_{\gamma, \gamma'} e^{-iE_{\gamma}t} \delta_{\gamma\gamma'}, \quad (5)$$

$$A_l^{\gamma\gamma'}(t) = \sum_{\gamma_1, \dots, \gamma_{l+1}} \left[\sum_{i=1}^{l+1} (-1)^{i-1} \frac{e^{-iE_{\gamma_i}t}}{d_i(E[\gamma, l])} \right] \left[\prod_{j=1}^l H_1^{\gamma_j\gamma_{j+1}} \right] \delta_{\gamma_1\gamma} \delta_{\gamma_{l+1}\gamma'}. \quad (6)$$

and all $H_1^{\gamma_j\gamma_{j+1}} = \langle \Phi^{\gamma_j} | H_1 | \Phi^{\gamma_{j+1}} \rangle$ form so-called “perturbing Hamiltonian matrix”, that is, the representation matrix of the perturbing Hamiltonian in the unperturbed Hamiltonian representation. While

$$d_1(E[\gamma, l]) = \prod_{i=1}^l (E_{\gamma_1} - E_{\gamma_{i+1}}), \quad (7)$$

$$d_i(E[\gamma, l]) = \prod_{j=1}^{i-1} (E_{\gamma_j} - E_{\gamma_i}) \prod_{k=i+1}^{l+1} (E_{\gamma_i} - E_{\gamma_k}), \quad (8)$$

$$d_{l+1}(E[\gamma, l]) = \prod_{i=1}^l (E_{\gamma_i} - E_{\gamma_{l+1}}), \quad (9)$$

here $2 \leq i \leq l$.

It is clear that there are apparent divergences in the above solution. For example

$$A_1^{\gamma\gamma'}(t) = \left[\frac{e^{-iE_{\gamma}t}}{E_{\gamma} - E_{\gamma'}} - \frac{e^{-iE_{\gamma'}t}}{E_{\gamma} - E_{\gamma'}} \right] H_1^{\gamma\gamma'}. \quad (10)$$

when $E_\gamma = E_{\gamma'}$ (which can appear in the summation or degeneracy cases), it is $\infty - \infty = -iH_1^{\gamma\gamma'}t$. As pointed out in our paper [3], we need to understand $A_k^{\gamma\gamma'}(t)$ in the sense of limitation. Moreover, in practice, we should present how to calculate their limitation in order to eliminate the apparent divergences.

Now, the key problems are how and when to introduce the cut-off approximation in order to obtain the perturbed solution. For studying and solving them, we first need to propose some skills and methods in this section. These skills and methods profit from the fact that the general term is clearly known and explicitly expressed in our exact solution. By using them we can derive out the improved forms of perturbed solution absorbing the partial contributions from the high order even all order approximations of perturbation. It will be seen that all the steps are well-regulated and only calculational technology is to find the limitation of primary functions. In other words, our exact solution and perturbation theory are easily calculative and operational, and they have better precision and higher efficiency. Frankly speaking, before we know our exact solution, we are puzzled by too many irregular terms and very trouble dependence on previous calculation steps. Moreover, we are often anxious about the precision of the results in such some calculations because those terms proportional to $t^a e^{-iE_{\gamma_i}t}$ ($a = 1, 2, \dots$) in the high order approximations might not be ignorable with time increasing. Considering the contributions of these terms can obviously improve the precision. However, the known perturbation theory does not give the general term, considering this task to absorb reasonably the high order approximations is impossible.

Since our exact solution has given the explicit form of any order approximation, that is a general term of an arbitrary order perturbed solution, and their forms are simply the summations of a power series of the perturbing Hamiltonian. Just enlightened by this general term of arbitrary order perturbed solution, we use two skills and “dynamical rearrangement and summation” method to build an improved scheme of perturbation theory, which are respectively expressed in the following two subsections.

A. Hamiltonian Redivision

The first skill starts from the decomposition of the perturbing Hamiltonian matrix, that is the matrix elements of H_1 in the representation of H_0 , into diagonal part and off-diagonal part:

$$H_1^{\gamma_j\gamma_{j+1}} = h_1^{\gamma_j}\delta_{\gamma_j\gamma_{j+1}} + g_1^{\gamma_j\gamma_{j+1}}, \quad (11)$$

so as to distinguish them because the diagonal and off-diagonal elements can be dealt with in the different way. In addition, it makes the concrete expression of a given order approximation can be easily calculated. Note that $h_1^{\gamma_j}$ has been chosen as its diagonal elements and then $g_1^{\gamma_j\gamma_{j+1}}$ has been set as its off-diagonal elements:

$$g_1^{\gamma_j\gamma_{j+1}} = g_1^{\gamma_j\gamma_{j+1}}(1 - \delta_{\gamma_j\gamma_{j+1}}). \quad (12)$$

As examples, for the first order approximation, it is easy to calculate that

$$A_1^{\gamma\gamma'}(h) = \sum_{\gamma_1, \gamma_2} \left[\sum_{i=1}^2 (-1)^{i-1} \frac{e^{-iE_{\gamma_i}t}}{d_i(E[\gamma, l])} \right] (h_1^{\gamma_1}\delta_{\gamma_1\gamma_2})\delta_{\gamma\gamma_1}\delta_{\gamma'\gamma_2} = \frac{(-ih_1^{\gamma}t)}{1!}e^{-iE_{\gamma}t}\delta_{\gamma\gamma'}, \quad (13)$$

$$A_1^{\gamma\gamma'}(g) = \sum_{\gamma_1, \gamma_2} \left[\sum_{i=1}^2 (-1)^{i-1} \frac{e^{-iE_{\gamma_i}t}}{d_i(E[\gamma, l])} \right] g_1^{\gamma_1\gamma_2}(1 - \delta_{\gamma_1\gamma_2})\delta_{\gamma\gamma_1}\delta_{\gamma'\gamma_2} = \left[\frac{e^{-iE_{\gamma}t}}{E_{\gamma} - E_{\gamma'}} - \frac{e^{-iE_{\gamma'}t}}{E_{\gamma} - E_{\gamma'}} \right] g_1^{\gamma\gamma'}. \quad (14)$$

Note that here and after we use the symbol $A_i^{\gamma\gamma'}$ denoting the contribution from the i th order approximation, which is defined by (6), while its argument indicates the product form of perturbing Hamiltonian matrix. However, for the second order approximation, since

$$\prod_{j=1}^2 H_1^{\gamma_j\gamma_{j+1}} = (h_1^{\gamma_1})^2 \delta_{\gamma_1\gamma_2}\delta_{\gamma_2\gamma_3} + h_1^{\gamma_1}g_1^{\gamma_2\gamma_3}\delta_{\gamma_1\gamma_2} + g_1^{\gamma_1\gamma_2}h_1^{\gamma_2}\delta_{\gamma_2\gamma_3} + g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma_3}. \quad (15)$$

we need to calculate the mixed product of diagonal and off-diagonal elements of perturbing Hamiltonian matrix. Obviously, we have

$$A_2^{\gamma\gamma'}(hh) = \sum_{\gamma_1, \gamma_2, \gamma_3} \left[\sum_{i=1}^3 (-1)^{i-1} \frac{e^{-iE_{\gamma_i}t}}{d_i(E[\gamma, 2])} \right] (h_1^{\gamma_1}\delta_{\gamma_1\gamma_2}h_1^{\gamma_2}\delta_{\gamma_2\gamma_3})\delta_{\gamma\gamma_1}\delta_{\gamma'\gamma_3} = \frac{(-ih_1^{\gamma}t)^2}{2!}e^{-iE_{\gamma}t}\delta_{\gamma\gamma'}, \quad (16)$$

$$\begin{aligned}
A_2^{\gamma\gamma'}(hg) &= \sum_{\gamma_1, \gamma_2, \gamma_3} \left[\sum_{i=1}^3 (-1)^{i-1} \frac{e^{-iE_{\gamma_i}t}}{d_i(E[\gamma, 2])} \right] h_1^{\gamma_1} g_1^{\gamma_2\gamma_3} \delta_{\gamma_1\gamma_2} \delta_{\gamma_1\gamma_3} \delta_{\gamma'\gamma_3} \\
&= \left[-\frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma'})^2} + \frac{e^{-iE_{\gamma'}t}}{(E_{\gamma} - E_{\gamma'})^2} + (-it) \frac{e^{-iE_{\gamma}t}}{E_{\gamma} - E_{\gamma'}} \right] h_1^{\gamma} g_1^{\gamma\gamma'}, \tag{17}
\end{aligned}$$

$$\begin{aligned}
A_2^{\gamma\gamma'}(gh) &= \sum_{\gamma_1, \gamma_2, \gamma_3} \left[\sum_{i=1}^3 (-1)^{i-1} \frac{e^{-iE_{\gamma_i}t}}{d_i(E[\gamma, 2])} \right] h_1^{\gamma_2} g_1^{\gamma_1\gamma_2} \delta_{\gamma_2\gamma_3} \delta_{\gamma_1\gamma_3} \delta_{\gamma'\gamma_3} \\
&= \left[\frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma'})^2} - \frac{e^{-iE_{\gamma'}t}}{(E_{\gamma} - E_{\gamma'})^2} - (-it) \frac{e^{-iE_{\gamma'}t}}{E_{\gamma} - E_{\gamma'}} \right] g_1^{\gamma\gamma'} h_1^{\gamma'}, \tag{18}
\end{aligned}$$

$$\begin{aligned}
A_2^{\gamma\gamma'}(gg) &= \sum_{\gamma_1, \gamma_2, \gamma_3} \left[\sum_{i=1}^3 (-1)^{i-1} \frac{e^{-iE_{\gamma_i}t}}{d_i(E[\gamma, 2])} \right] g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} \delta_{\gamma_1\gamma_3} \delta_{\gamma'\gamma_3} \\
&= \sum_{\gamma_1} \left[\frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma'})} - \frac{e^{-iE_{\gamma_1}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma'})} + \frac{e^{-iE_{\gamma'}t}}{(E_{\gamma} - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})} \right] g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}. \tag{19}
\end{aligned}$$

In the usual time-dependent perturbation theory, the zeroth order approximation of time evolution of quantum state keeps its original form

$$|\Psi^{(0)}(t)\rangle = e^{-iE_{\gamma}t} |\Phi^{\gamma}\rangle, \tag{20}$$

where we have set the initial state as $|\Phi^{\gamma}\rangle$ for simplicity. By using our solution, we easily calculate out the contributions of all order approximations from the product of completely diagonal elements h of the perturbing Hamiltonian matrix to this zeroth order approximation

$$\sum_{\gamma_1, \dots, \gamma_{l+1}} \left[\sum_{i=1}^{l+1} (-1)^{i-1} \frac{e^{-iE_{\gamma_i}t}}{d_i(E[\gamma, l])} \right] \left(\prod_{j=1}^l h_1^{\gamma_j} \delta_{\gamma_j\gamma_{j+1}} \right) \delta_{\gamma\gamma_1} \delta_{\gamma'\gamma_{l+1}} = \frac{(-ih_1^{\gamma}t)^l}{l!} e^{-iE_{\gamma}t} \delta_{\gamma\gamma'}. \tag{21}$$

Therefore, we can absorb the contributions of all order approximation parts from the product of completely diagonal elements h of the perturbing Hamiltonian matrix into this zeroth order approximation to obtain

$$|\Psi'^{(0)}(t)\rangle = e^{-i(E_{\gamma} + h_1^{\gamma})t} |\Phi^{\gamma}\rangle. \tag{22}$$

Similarly, by calculation, we can deduce that up to the second approximation, the perturbed solution has the following form

$$\begin{aligned}
|\Psi'(t)\rangle &= \sum_{\gamma, \gamma'} \left\{ e^{-i(E_{\gamma} + h_1^{\gamma})t} \delta_{\gamma\gamma'} + \left[\frac{e^{-i(E_{\gamma} + h_1^{\gamma})t} - e^{-i(E_{\gamma'} + h_1^{\gamma'})t}}{(E_{\gamma} + h_1^{\gamma}) - (E_{\gamma'} + h_1^{\gamma'})} \right] g_1^{\gamma\gamma'} \right. \\
&\quad + \sum_{\gamma_1} \left[\frac{e^{-i(E_{\gamma} + h_1^{\gamma})t}}{[(E_{\gamma} + h_1^{\gamma}) - (E_{\gamma_1} + h_1^{\gamma_1})] [(E_{\gamma} + h_1^{\gamma}) - (E_{\gamma'} + h_1^{\gamma'})]} \right. \\
&\quad - \frac{e^{-i(E_{\gamma_1} + h_1^{\gamma_1})t}}{[(E_{\gamma} + h_1^{\gamma}) - (E_{\gamma_1} + h_1^{\gamma_1})] [(E_{\gamma_1} + h_1^{\gamma_1}) - (E_{\gamma'} + h_1^{\gamma'})]} \\
&\quad \left. \left. + \frac{e^{-i(E_{\gamma'} + h_1^{\gamma'})t}}{[(E_{\gamma} + h_1^{\gamma}) - (E_{\gamma'} + h_1^{\gamma'})] [(E_{\gamma_1} + h_1^{\gamma_1}) - (E_{\gamma'} + h_1^{\gamma'})]} \right] g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \right\} \\
&\quad [\langle \Phi^{\gamma'} | \Psi(0) \rangle] |\Phi^{\gamma}\rangle + \mathcal{O}(H_1^3). \tag{23}
\end{aligned}$$

However, for the higher order approximation, the corresponding calculation is heavy. In fact, it is unnecessary to calculate the contributions from those terms with the diagonal elements of H_1 since introducing the following skill.

This is a reason why we omit the relevant calculation details. Here we mention it only for verification of the correctness of our exact solution in this way.

The results (22) and (23) are not surprised because of the fact that the Hamiltonian is re-divisible. Actually, we can furthermore use a trick of redivision of the Hamiltonian so that the new H_0 contains the diagonal part of H_1 , that is,

$$H'_0 = H_0 + \sum_{\gamma} h_1^{\gamma} |\Phi^{\gamma}\rangle \langle \Phi^{\gamma}|, \quad (24)$$

$$H'_1 = H_1 - \sum_{\gamma} h_1^{\gamma} |\Phi^{\gamma}\rangle \langle \Phi^{\gamma}| = \sum_{\gamma, \gamma'} g_1^{\gamma\gamma'} |\Phi^{\gamma}\rangle \langle \Phi^{\gamma'}|. \quad (25)$$

In other words, without loss of generality, we always can choose that H'_1 has only the off-diagonal elements in the H'_0 (or H_0) representation and

$$H'_0 |\Phi^{\gamma}\rangle = (E_{\gamma} + h_1^{\gamma}) |\Phi^{\gamma}\rangle = E'_{\gamma} |\Phi^{\gamma}\rangle. \quad (26)$$

It is clear that this redivision does not change the representation of the unperturbed Hamiltonian, but can change the corresponding eigenvalues. In spite that our skill is so simple, it seems not be sufficiently transpired and understood from the fact that the recent some textbooks of quantum mechanics still remain the contributions from the diagonal elements of the perturbing Hamiltonian matrix in the expression of the second order perturbed state. It is clear that the directly cut-off approximation in the usual perturbation theory drops the contributions from all higher order approximations of the diagonal element of the perturbing Hamiltonian matrix. From our point of view, the usual perturbation theory introduces the perturbing parameter too early so that this flaw is resulted in.

If there is degeneracy, our notation has to be changed as

$$E_{\gamma_i} \rightarrow E_{\gamma_i a_{\gamma_i}} = E_{\gamma_i}, \quad (27)$$

$$\delta_{\gamma_i \gamma_j} \rightarrow \delta_{\gamma_i \gamma_j} \delta_{a_{\gamma_i} a_{\gamma_j}}, \quad (28)$$

$$\eta_{\gamma_i \gamma_j} \rightarrow \eta_{\gamma_i \gamma_j} + \delta_{\gamma_i \gamma_j} \eta_{a_{\gamma_i} a_{\gamma_j}}. \quad (29)$$

Thus, we can find

$$A_1^{\gamma a_{\gamma}, \gamma' a_{\gamma'}}(h) = \frac{(-i h_1^{\gamma} t)}{1!} e^{-i E_{\gamma} t} \delta_{\gamma \gamma'} \delta_{a_{\gamma} a_{\gamma'}}, \quad (30)$$

$$A_1^{\gamma a_{\gamma}, \gamma' a_{\gamma'}}(g) = \left[\frac{e^{-i E_{\gamma a_{\gamma}} t}}{E_{\gamma a_{\gamma}} - E_{\gamma' a_{\gamma'}}} - \frac{e^{-i E_{\gamma' a_{\gamma'}} t}}{E_{\gamma a_{\gamma}} - E_{\gamma' a_{\gamma'}}} \right] g_1^{\gamma a_{\gamma}, \gamma' a_{\gamma'}} \eta_{\gamma \gamma'} + \frac{(-i g_1^{\gamma a_{\gamma}, \gamma' a_{\gamma'}} t)}{1!} e^{-i E_{\gamma a_{\gamma}} t} \delta_{\gamma \gamma'}. \quad (31)$$

This seems to bring some complications. However, we can use the trick in the usual degenerate perturbation theory, that is, we are free to choose our base set of unperturbed kets $|\Phi^{\gamma a_{\gamma}}\rangle$ in such a way that that H_1 is diagonalized in the corresponding degenerate subspaces. In other words, we should find the linear combinations of the degenerate unperturbed kets to re-span the zero-order eigen subspace of H_0 so that

$$\langle \Phi^{\gamma a_{\gamma}} | H_1 | \Phi^{\gamma b_{\gamma}} \rangle = g_1^{\gamma a_{\gamma}, \gamma b_{\gamma}} = d_1^{\gamma a_{\gamma}} \delta_{a_{\gamma} b_{\gamma}}. \quad (32)$$

(If there is still the same values among all of $d_{\gamma a}$, this procedure can be repeated in general.) This means that $g_1^{\gamma a_{\gamma}, \gamma' a_{\gamma'}} = 0$. Then, we use our redivision skill again, that is

$$H''_0 = H_0 + \sum_{\gamma \notin D, \gamma} h_1^{\gamma} |\Phi^{\gamma}\rangle \langle \Phi^{\gamma}| + \sum_{\gamma \in D, a_{\gamma}} d_1^{\gamma a_{\gamma}} |\Phi^{\gamma a_{\gamma}}\rangle \langle \Phi^{\gamma a_{\gamma}}|, \quad (33)$$

$$H''_1 = H_1 - \sum_{\gamma \notin D, \gamma} h_1^{\gamma} |\Phi^{\gamma}\rangle \langle \Phi^{\gamma}| - \sum_{\gamma \in D, a_{\gamma}} d_1^{\gamma a_{\gamma}} |\Phi^{\gamma a_{\gamma}}\rangle \langle \Phi^{\gamma a_{\gamma}}|. \quad (34)$$

where D is a set of all degenerate subspace-indexes. Thus, the last term in Eq. (31) vanishes,

$$A_1^{\gamma a_{\gamma}, \gamma' a_{\gamma'}}(g) = \left[\frac{e^{-i E_{\gamma a_{\gamma}} t}}{E_{\gamma a_{\gamma}} - E_{\gamma' a_{\gamma'}}} - \frac{e^{-i E_{\gamma' a_{\gamma'}} t}}{E_{\gamma a_{\gamma}} - E_{\gamma' a_{\gamma'}}} \right] g_1^{\gamma a_{\gamma}, \gamma' a_{\gamma'}} \eta_{\gamma \gamma'}. \quad (35)$$

In fact, under the preconditions of H_1 is diagonal in the degenerate subspaces, we can directly do replacement

$$g_1^{\gamma_i \gamma_j} \rightarrow g_1^{\gamma_i a_{\gamma_i}, \gamma_j a_{\gamma_j}} \eta_{\gamma_i \gamma_j} \quad (36)$$

from the non-degenerate case to the degenerate case. For simplicity, we always assume that H_1 has been diagonalized in the degenerate subspaces from now on.

It must be emphasized that the Hamiltonian redivision skill leads to the fact that the new perturbed solution can be obtained by the replacement

$$E_{\gamma_i} \rightarrow E_{\gamma_i} + h_1^{\gamma_i} \quad (37)$$

in the non-degenerate perturbed solution and its conclusions. With degeneracy present, if our method is to work well, the degeneracy should be completely removed in the diagonalization procedures of the degenerate subspaces and the Hamiltonian redivision, that is, for any given degenerate subspace, $d_1^a \neq d_1^b$ if $a \neq b$. In other words, $E''_{\gamma_a} \neq E''_{\gamma_b}$ if $a \neq b$. This means that all of eigenvalues of H_0'' are no longer the same, so we can back to the non-degenerate cases. Or specially, if we allow the remained degeneracies, the off-diagonal element of the perturbing Hamiltonian matrix between any two degenerate levels are always vanishing. This implies that there is no extra contribution from the degeneracies in the any more than the zeroth order approximations. It is important to remember these facts. However, how must we proceed if the degeneracies are not completely removed by the usual diagonalization procedure and our Hamiltonian redivision as well as the special cases with the remained degeneracies stated above are not valid. It is known to be a challenge in the usual perturbation theory. Although our exact solution can apply to such a kind of cases, but the form of perturbed solution will get complicated because more apparent divergences need to be eliminated and then some new terms proportional to the power of evolution time will appear in general. We will study this problem in the near future. Based on the above reasons, we do not consider the degenerate case from now on.

From the statement above, we have seen that there are two equivalent ways to obtain the same perturbed solution and its conclusions. One of them is to redefine the energy level E_{γ_i} as E'_{γ_i} (or E''_{γ_i}), think E'_{γ_i} (or E''_{γ_i}) to be explicitly independent on the perturbing parameter from a redefined view, and then use the method in the usual perturbation theory to obtain the result from the redivided H_1' (or H_1''). The other way is to directly derive out the perturbed solution from the original Hamiltonian by using the standard procedure, but the rearrangement and summation are carried out just like what we have done above. From our point of view, this is because the perturbing parameter is only a formal multiplier in mathematics and it can be introduced after redefining E'_{γ_i} . It is natural although this problem seems not be noticed for a long time. The first skill, that is, the Hamiltonian redivision skill will be again applied to our scheme to obtain improved forms of perturbed energy and perturbed state in Sec. III.

The Hamiltonian redivision not only overcomes the flaw of the usual perturbation theory, but also has three obvious advantages. Firstly, it advances the calculation precision of perturbation theory because it makes the contributions from all order approximations of the diagonal elements of the perturbing Hamiltonian matrix are naturally included. Secondly, it extends the applicable range of perturbation theory based on the same reason since the diagonal elements of the perturbing Hamiltonian no longer is needed to be smaller. Lastly, it can be used to remove the degeneracies, which is important for the perturbation theory.

For simplicity, in the following, we omit the ' (or '') in H_0 , H_1 as well as E_{γ} , and always let H_1 have only its off-diagonal part and let H_0 have no degeneracy unless particular claiming.

B. Perturbed Hamiltonian matrix product decomposition and apparent divergence elimination

In this subsection, we present the second important skill enlightened by our exact solution, that is, the perturbing Hamiltonian matrix product decomposition, which is a technology to separate the contraction terms with apparent divergences and anti-contraction terms without apparent divergences, and then we can eliminate these apparent divergences by the limit process. More importantly, we can propose so-called ‘‘dynamical rearrangement and summation’’ idea in order to absorb the partial contributions from the high order even all order approximation of perturbing Hamiltonian into the lower order terms of our perturbation theory. It is a key method in our improved scheme of perturbation theory.

Let us start with the second order approximation. Since we have taken $H_1^{\gamma_j \gamma_{j+1}}$ only with the off-diagonal part $g_1^{\gamma_j \gamma_{j+1}}$, the contribution from the second order approximation of the perturbing Hamiltonian is only $A_2^{\gamma \gamma'}(gg)$ in eq.(19). However, we find that in the above expression of $A_2^{\gamma \gamma'}(gg)$, the apparent divergence has not been completely eliminated or the limitation has not been completely found out because we have not excluded the case $E_{\gamma} = E_{\gamma'}$ (or $\gamma = \gamma'$). This problem can be fixed by introducing a perturbing Hamiltonian matrix product decomposition

$$g_1^{\gamma_1 \gamma_2} g_1^{\gamma_2 \gamma_3} = g_1^{\gamma_1 \gamma_2} g_1^{\gamma_2 \gamma_3} \delta_{\gamma_1 \gamma_3} + g_1^{\gamma_1 \gamma_2} g_1^{\gamma_2 \gamma_3} \eta_{\gamma_1 \gamma_3}, \quad (38)$$

where $\eta_{\gamma_1\gamma_3} = 1 - \delta_{\gamma_1\gamma_3}$. Thus, the contribution from the second order approximation is made of two terms, one so-called contraction term with the δ function factor and the other so-called anti-contraction term with the η function factor. Obviously, the contraction term has the apparent divergence and anti-contraction term has no the apparent divergence. Hence, in order to eliminate the apparent divergence in the contraction term, we only need to find its limitation. It must be emphasized that we only consider the non-degenerate case here and after for simplification. When the degeneration happens, two indexes with the same main energy level number will not have the anti-contraction.

In terms of the above skill, we find that the contribution from the second order approximation is made of the corresponding contraction- and anti-contraction- terms

$$A_2^{\gamma\gamma'}(gg) = A_2^{\gamma\gamma'}(gg; c) + A_2^{\gamma\gamma'}(gg; n), \quad (39)$$

where

$$\begin{aligned} A_2^{\gamma\gamma'}(gg; c) &= \sum_{\gamma_1, \gamma_2, \gamma_3} \left[\sum_{i=1}^3 (-1)^{i-1} \frac{e^{-iE_{\gamma_i}t}}{d_i(E[\gamma, 2])} \right] g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} \delta_{\gamma_1\gamma_3} \delta_{\gamma_1\gamma'} \delta_{\gamma'\gamma_3} \\ &= \sum_{\gamma_1} \left[-\frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})^2} + \frac{e^{-iE_{\gamma_1}t}}{(E_{\gamma} - E_{\gamma_1})^2} + (-it) \frac{e^{-iE_{\gamma}t}}{E_{\gamma} - E_{\gamma_1}} \right] |g_1^{\gamma\gamma_1}|^2 \delta_{\gamma\gamma'}, \end{aligned} \quad (40)$$

$$\begin{aligned} A_2^{\gamma\gamma'}(gg; n) &= \sum_{\gamma_1, \gamma_2, \gamma_3} \left[\sum_{i=1}^3 (-1)^{i-1} \frac{e^{-iE_{\gamma_i}t}}{d_i(E[\gamma, 2])} \right] g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} \eta_{\gamma_1\gamma_3} \delta_{\gamma_1\gamma'} \delta_{\gamma'\gamma_3} \\ &= \sum_{\gamma_1} \left[\frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma'})} - \frac{e^{-iE_{\gamma_1}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma'})} + \frac{e^{-iE_{\gamma'}t}}{(E_{\gamma} - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})} \right] g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'}. \end{aligned} \quad (41)$$

The above method can be extended to the higher order approximation by introducing a skill of perturbing Hamiltonian matrix product decomposition, or simply called it g -product decomposition when the perturbing Hamiltonian matrix is off-diagonal. For a sequential product of off-diagonal elements g with the form $\prod_{k=1}^m g_1^{\gamma_k\gamma_{k+1}}$ ($m \geq 2$), we define its $(m-1)$ th decomposition by

$$\prod_{k=1}^m g_1^{\gamma_k\gamma_{k+1}} = \left(\prod_{k=1}^m g_1^{\gamma_k\gamma_{k+1}} \right) \delta_{\gamma_1\gamma_{m+1}} + \left(\prod_{k=1}^m g_1^{\gamma_k\gamma_{k+1}} \right) \eta_{\gamma_1\gamma_{m+1}}. \quad (42)$$

When we calculate the contributions from the n th order approximation, we will first carry out $(n-1)$ the first decompositions, that is

$$\prod_{k=1}^n g_1^{\gamma_k\gamma_{k+1}} = \left(\prod_{k=1}^n g_1^{\gamma_k\gamma_{k+1}} \right) \left[\prod_{k=1}^{n-1} (\delta_{\gamma_k\gamma_{k+2}} + \eta_{\gamma_k\gamma_{k+2}}) \right]. \quad (43)$$

Obviously, from the fact that H_1 is usually taken as Hermit one, it follows that

$$g_1^{\gamma_j\gamma_{j+1}} g_1^{\gamma_{j+1}\gamma_{j+2}} \delta_{\gamma_j\gamma_{j+2}} = |g_1^{\gamma_j\gamma_{j+1}}|^2 \delta_{\gamma_j\gamma_{j+2}}. \quad (44)$$

When the contribution from a given order approximation is considered, the summation over one of two subscripts will lead to the contraction of g -production. More generally, for the contraction of even number g -production

$$\left(\prod_{j=1}^m g_1^{\gamma_j\gamma_{j+1}} \prod_{k=1}^{m-1} \delta_{\gamma_k\gamma_{k+2}} \right) \delta_{\gamma_1\gamma} \delta_{\gamma_{m+1}\gamma'} = |g_1^{\gamma\gamma_2}|^m \left(\prod_{k=1}^{m-1} \delta_{\gamma_k\gamma_{k+2}} \right) \delta_{\gamma_1\gamma} \delta_{\gamma_{m+1}\gamma'} \delta_{\gamma\gamma'}, \quad (45)$$

and for the contraction of odd number g -production,

$$\left(\prod_{j=1}^m g_1^{\gamma_j\gamma_{j+1}} \prod_{k=1}^{m-1} \delta_{\gamma_k\gamma_{k+2}} \right) \delta_{\gamma_1\gamma} \delta_{\gamma_{m+1}\gamma'} = |g_1^{\gamma\gamma'}|^{m-1} \left(\prod_{k=1}^{m-1} \delta_{\gamma_k\gamma_{k+2}} \right) \delta_{\gamma_1\gamma} \delta_{\gamma_{m+1}\gamma'} g_1^{\gamma\gamma'}, \quad (46)$$

where $\delta_{\gamma_1\gamma} \delta_{\gamma_{m+1}\gamma'}$ is a factor appearing in the expression of our solution.

Then, we consider, in turn, all possible the second decomposition, the third decomposition, and up to the $(n-1)$ th decomposition. It must be emphasized that after calculating the contributions from the terms of lower decompositions, some of terms in the higher decompositions may be trivial because there are some symmetric and complementary symmetric indexes in the corresponding results, that is, the products of these results and the given $\delta_{\gamma_k \gamma_{k'}}$ or $\eta_{\gamma_k \gamma_{k'}}$ are zero. In other words, such some higher decompositions do not need to be considered. As an example, let us analyze the contribution from the third order approximation. It is clear that the first decomposition of a sequential production of three off-diagonal elements becomes

$$\begin{aligned} g_1^{\gamma_1 \gamma_2} g_1^{\gamma_2 \gamma_3} g_1^{\gamma_3 \gamma_4} &= g_1^{\gamma_1 \gamma_2} g_1^{\gamma_2 \gamma_3} g_1^{\gamma_3 \gamma_4} \delta_{\gamma_1 \gamma_3} \delta_{\gamma_2 \gamma_4} + g_1^{\gamma_1 \gamma_2} g_1^{\gamma_2 \gamma_3} g_1^{\gamma_3 \gamma_4} \delta_{\gamma_1 \gamma_3} \eta_{\gamma_2 \gamma_4} \\ &\quad + g_1^{\gamma_1 \gamma_2} g_1^{\gamma_2 \gamma_3} g_1^{\gamma_3 \gamma_4} \eta_{\gamma_1 \gamma_3} \delta_{\gamma_2 \gamma_4} + g_1^{\gamma_1 \gamma_2} g_1^{\gamma_2 \gamma_3} g_1^{\gamma_3 \gamma_4} \eta_{\gamma_1 \gamma_3} \eta_{\gamma_2 \gamma_4}. \end{aligned} \quad (47)$$

Thus, the related contribution is just divided into 4 terms

$$A_3^{\gamma \gamma'}(ggg) = A_3^{\gamma \gamma'}(ggg; cc) + A_3^{\gamma \gamma'}(ggg; cn) + A_3^{\gamma \gamma'}(ggg; nc) + A_3^{\gamma \gamma'}(ggg; nn). \quad (48)$$

In fact, by calculating we know that the second decomposition of the former three terms do not need to be considered, only the second decomposition of the last term is nontrivial. This means that

$$A_3^{\gamma \gamma'}(ggg; nn) = A_3^{\gamma \gamma'}(ggg; nn, c) + A_3^{\gamma \gamma'}(ggg; nn, n), \quad (49)$$

where we have added $\delta_{\gamma_1 \gamma_3}$ in the definition of $A_3^{\gamma \gamma'}(ggg; nn, c)$, and $\eta_{\gamma_1 \gamma_3}$ in the definition of $A_3^{\gamma \gamma'}(ggg; nn, n)$. Obviously, in the practical process, this feature largely simplifies the calculations. It is easy to see that the number of all of terms with contractions and anti-contractions is 5. For convenience and clearness, we call the contributions from the different terms in the decomposition of g -product as the contractions and anti-contractions of g -product. Of course, the contraction and anti-contraction refer to the meaning after summation(s) over the subscript(s) in general. Moreover, here and after, we drop the argument $gg \cdots g$ in the i th order approximation A_i since its meaning has been indicated by i after the Hamiltonian is redivided. For example, the explicit expressions of all contraction- and anti-contraction terms in the third order approximation A_3 can be calculated as follows:

$$\begin{aligned} A_3^{\gamma \gamma'}(cc) &= \sum_{\gamma_1, \dots, \gamma_4} \left[\sum_{i=1}^4 (-1)^{i-1} \frac{e^{-iE_{\gamma_i} t}}{d_i(E[\gamma, 3])} \right] \left[\prod_{j=1}^3 g_1^{\gamma_j \gamma_{j+1}} \right] \left(\prod_{k=1}^2 \delta_{\gamma_k \gamma_{k+2}} \right) \delta_{\gamma_1 \gamma} \delta_{\gamma_4 \gamma'} \\ &= \left[-\frac{2e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma'})^3} + \frac{2e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'})^3} + (-it) \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma'})^2} + (-it) \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'})^2} \right] |g_1^{\gamma \gamma'}|^2 g_1^{\gamma \gamma'}, \end{aligned} \quad (50)$$

$$\begin{aligned} A_3^{\gamma \gamma'}(cn) &= \sum_{\gamma_1, \dots, \gamma_4} \left[\sum_{i=1}^4 (-1)^{i-1} \frac{e^{-iE_{\gamma_i} t}}{d_i(E[\gamma, 3])} \right] \left[\prod_{j=1}^3 g_1^{\gamma_j \gamma_{j+1}} \right] \delta_{\gamma_1 \gamma_3} \eta_{\gamma_2 \gamma_4} \delta_{\gamma_1 \gamma} \delta_{\gamma_{l+1} \gamma'} \\ &= \sum_{\gamma_1} \left[-\frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma'})^2} - \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^2(E_{\gamma} - E_{\gamma'})} + \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})^2(E_{\gamma_1} - E_{\gamma'})} \right. \\ &\quad \left. - \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'})^2(E_{\gamma_1} - E_{\gamma'})} + (-it) \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma'})} \right] |g_1^{\gamma \gamma_1}|^2 g_1^{\gamma \gamma'} \eta_{\gamma_1 \gamma'}, \end{aligned} \quad (51)$$

$$\begin{aligned} A_3^{\gamma \gamma'}(nc) &= \sum_{\gamma_1, \dots, \gamma_4} \left[\sum_{i=1}^4 (-1)^{i-1} \frac{e^{-iE_{\gamma_i} t}}{d_i(E[\gamma, 3])} \right] \left[\prod_{j=1}^3 g_1^{\gamma_j \gamma_{j+1}} \right] \eta_{\gamma_1 \gamma_3} \delta_{\gamma_2 \gamma_4} \delta_{\gamma_1 \gamma} \delta_{\gamma_{l+1} \gamma'} \\ &= \sum_{\gamma_1} \left[\frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma'})^2} - \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma'})^2} - \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma'})^2} \right. \\ &\quad \left. + \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma'})^2} + (-it) \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'})^2(E_{\gamma_1} - E_{\gamma'})} \right] g_1^{\gamma \gamma'} |g_1^{\gamma_1 \gamma'}|^2 \eta_{\gamma \gamma_1}, \end{aligned} \quad (52)$$

$$\begin{aligned}
A_3^{\gamma\gamma'}(nn, c) &= \sum_{\gamma_1, \dots, \gamma_4} \left[\sum_{i=1}^4 (-1)^{i-1} \frac{e^{-iE_{\gamma_i} t}}{d_i(E[\gamma, 3])} \right] \left[\prod_{j=1}^3 g_1^{\gamma_j \gamma_{j+1}} \right] \delta_{\gamma_1 \gamma} \delta_{\gamma_4 \gamma'} \eta_{\gamma_1 \gamma_3} \eta_{\gamma_2 \gamma_4} \delta_{\gamma \gamma'} \\
&= \sum_{\gamma_1 \gamma_2} \left[-\frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2})} + \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_2})^2 (E_{\gamma_1} - E_{\gamma_2})} + \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2})} \right. \\
&\quad \left. - \frac{e^{-iE_{\gamma_2} t}}{(E_{\gamma} - E_{\gamma_2})^2 (E_{\gamma_1} - E_{\gamma_2})} + (-it) \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1}) (E_{\gamma_1} - E_{\gamma_2})} \right] g_1^{\gamma \gamma_1} g_1^{\gamma_1 \gamma_2} g_1^{\gamma_2 \gamma'} \eta_{\gamma \gamma_2} \delta_{\gamma \gamma'}, \quad (53)
\end{aligned}$$

$$\begin{aligned}
A_3^{\gamma\gamma'}(nn, n) &= \sum_{\gamma_1, \dots, \gamma_4} \left[\sum_{i=1}^4 (-1)^{i-1} \frac{e^{-iE_{\gamma_i} t}}{d_i(E[\gamma, 3])} \right] \left[\prod_{j=1}^3 g_1^{\gamma_j \gamma_{j+1}} \right] \delta_{\gamma_1 \gamma} \delta_{\gamma_4 \gamma'} \eta_{\gamma_1 \gamma_3} \eta_{\gamma_2 \gamma_4} \eta_{\gamma \gamma'} \\
&= \sum_{\gamma_1 \gamma_2} \left[\frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1}) (E_{\gamma} - E_{\gamma_2}) (E_{\gamma} - E_{\gamma'})} - \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1}) (E_{\gamma_1} - E_{\gamma_2}) (E_{\gamma_1} - E_{\gamma'})} \right. \\
&\quad \left. + \frac{e^{-iE_{\gamma_2} t}}{(E_{\gamma} - E_{\gamma_2}) (E_{\gamma_1} - E_{\gamma_2}) (E_{\gamma_2} - E_{\gamma'})} - \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})} \right] \\
&\quad \times g_1^{\gamma \gamma_1} g_1^{\gamma_1 \gamma_2} g_1^{\gamma_2 \gamma'} \eta_{\gamma \gamma_2} \eta_{\gamma_1 \gamma'} \eta_{\gamma \gamma'}. \quad (54)
\end{aligned}$$

In the above calculations, the used technologies mainly to find the limitation, dummy index changing and summation, as well as the replacement $g_1^{\gamma_i \gamma_j} \eta_{\gamma_i \gamma_j} = g_1^{\gamma_i \gamma_j}$ since $g_1^{\gamma_i \gamma_j}$ has been off-diagonal.

It must be emphasized that, in our notation, $A_i^{\gamma\gamma'}$ represents the contributions from the i th order approximation. The other independent variables are divided into $i - 1$ groups and are arranged sequentially corresponding to the order of g -product decomposition. That is, the first variable group represents the first decomposition, the second variable group represents the second decomposition, and so on. Every variable group is a bit-string made of three possible element c, n, k and its length is equal to the number of the related order of g -product decomposition, that is, for the j th decompositions in the i th order approximation its length is $i - j$. In each variable group, c corresponds to a δ function, n corresponds to a η function and k corresponds to 1 (non-decomposition). Their sequence in the bit-string corresponds to the sequence of contraction and/or anti-contraction index string. From the above analysis and statement, the index string of the j th decompositions in the i order approximation is:

$$\prod_{k=1}^{i-j} (\gamma_k, \gamma_{k+1+j}). \quad (55)$$

For example, for A_5 , the first variable group is $cccn$, which refers to the first decomposition in five order approximation and the terms to include the factor $\delta_{\gamma_1 \gamma_3} \delta_{\gamma_2 \gamma_4} \delta_{\gamma_3 \gamma_5} \eta_{\gamma_4 \gamma_6}$ in the definition of $A_5(cccn)$. Similarly, $cncc$ means to insert the factor $\delta_{\gamma_1 \gamma_3} \eta_{\gamma_2 \gamma_4} \delta_{\gamma_3 \gamma_5} \delta_{\gamma_4 \gamma_6}$ into the definition of $A_5(cncc)$. When there are nontrivial second contractions, for instance, two variable groups ($ccnn, kkc$) represent that the definition of $A_5(ccnn, kkc)$ has the factor $(\delta_{\gamma_1 \gamma_3} \delta_{\gamma_2 \gamma_4} \eta_{\gamma_3 \gamma_5} \eta_{\gamma_4 \gamma_6}) \delta_{\gamma_3 \gamma_6}$. Since there are fully trivial contraction (the bit-string is made of only k), we omit their related variable group for simplicity.

Furthermore, we pack up all the contraction- and non-contraction terms in the following way so that we can obtain conveniently the improved forms of perturbed solution of dynamics absorbing the partial contributions from the high order even all order approximations. We first decompose $A_3^{\gamma\gamma'}$, which is a summation of all above terms, into the three parts according to $e^{-iE_{\gamma_i} t}$, $(-it)e^{-iE_{\gamma_i} t}$ and $(-it)^2 e^{-iE_{\gamma_i} t}/2$:

$$A_3^{\gamma\gamma'} = A_3^{\gamma\gamma'}(e) + A_3^{\gamma\gamma'}(te) + A_3^{\gamma\gamma'}(t^2e). \quad (56)$$

Secondly, we decompose its every term into three parts according to $e^{-iE_{\gamma} t}$, $e^{-iE_{\gamma_1} t}$ ($\sum_{\gamma_1} e^{-iE_{\gamma_1} t}$) and $e^{-iE_{\gamma'} t}$:

$$A_3^{\gamma\gamma'}(e) = A_3^{\gamma\gamma'}(e^{-iE_{\gamma} t}) + A_3^{\gamma\gamma'}(e^{-iE_{\gamma_1} t}) + A_3^{\gamma\gamma'}(e^{-iE_{\gamma'} t}), \quad (57)$$

$$A_3^{\gamma\gamma'}(te) = A_3^{\gamma\gamma'}(te^{-iE_{\gamma} t}) + A_3^{\gamma\gamma'}(te^{-iE_{\gamma_1} t}) + A_3^{\gamma\gamma'}(te^{-iE_{\gamma'} t}), \quad (58)$$

$$A_4^{\gamma\gamma'}(t^2e) = A_3^{\gamma\gamma'}(t^2e^{-iE_{\gamma} t}) + A_3^{\gamma\gamma'}(t^2e^{-iE_{\gamma_1} t}) + A_3^{\gamma\gamma'}(t^2e^{-iE_{\gamma'} t}). \quad (59)$$

Finally, we again decompose every term in the above equations into the diagonal and off-diagonal parts about γ and γ' :

$$A_3^{\gamma\gamma'}(e^{-iE_{\gamma_i}t}) = A_3^{\gamma\gamma'}(e^{-iE_{\gamma_i}t}; D) + A_3^{\gamma\gamma'}(e^{-iE_{\gamma_i}t}; N), \quad (60)$$

$$A_3^{\gamma\gamma'}(te^{-iE_{\gamma_i}t}) = A_3^{\gamma\gamma'}(te^{-iE_{\gamma_i}t}; D) + A_3^{\gamma\gamma'}(te^{-iE_{\gamma_i}t}; N), \quad (61)$$

$$A_3^{\gamma\gamma'}(t^2e^{-iE_{\gamma_i}t}) = A_3^{\gamma\gamma'}(t^2e^{-iE_{\gamma_i}t}; D) + A_3^{\gamma\gamma'}(t^2e^{-iE_{\gamma_i}t}; N), \quad (62)$$

where E_{γ_i} takes E_γ, E_{γ_1} and $E_{\gamma'}$.

According to the above way, it is easy to obtain

$$A_3^{\gamma\gamma'}(e^{-iE_\gamma t}; D) = - \sum_{\gamma_1, \gamma_2} e^{-iE_\gamma t} \left[\frac{1}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})^2} + \frac{1}{(E_\gamma - E_{\gamma_1})^2(E_\gamma - E_{\gamma_2})} \right] g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} \delta_{\gamma\gamma'}, \quad (63)$$

$$\begin{aligned} A_3^{\gamma\gamma'}(e^{-iE_\gamma t}; N) &= - \sum_{\gamma_1} e^{-iE_\gamma t} \left[\frac{1}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma'})^2} + \frac{1}{(E_\gamma - E_{\gamma_1})^2(E_\gamma - E_{\gamma'})} \right] g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'} \\ &\quad + \sum_{\gamma_1, \gamma_2} e^{-iE_\gamma t} \frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})}, \end{aligned} \quad (64)$$

$$A_3^{\gamma\gamma'}(e^{-iE_{\gamma_1}t}; D) = \sum_{\gamma_1, \gamma_2} e^{-iE_{\gamma_1}t} \frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} \delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})^2(E_{\gamma_1} - E_{\gamma_2})}, \quad (65)$$

$$A_3^{\gamma\gamma'}(e^{-iE_{\gamma_1}t}; N) = - \sum_{\gamma_1, \gamma_2} e^{-iE_{\gamma_1}t} \frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma_1\gamma'} \eta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma'})}, \quad (66)$$

$$A_3^{\gamma\gamma'}(e^{-iE_{\gamma_2}t}; D) = - \sum_{\gamma_1, \gamma_2} e^{-iE_{\gamma_2}t} \frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} \delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_2})^2(E_{\gamma_1} - E_{\gamma_2})}, \quad (67)$$

$$A_3^{\gamma\gamma'}(e^{-iE_{\gamma_2}t}; N) = \sum_{\gamma_1, \gamma_2} e^{-iE_{\gamma_2}t} \frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_2} - E_{\gamma'})}, \quad (68)$$

$$A_3^{\gamma\gamma'}(e^{-iE_{\gamma'}t}; D) = 0, \quad (69)$$

$$\begin{aligned} A_3^{\gamma\gamma'}(e^{-iE_{\gamma'}t}; N) &= \sum_{\gamma_1} e^{-iE_{\gamma'}t} \left[\frac{1}{(E_\gamma - E_{\gamma'})^2(E_{\gamma_1} - E_{\gamma'})} + \frac{1}{(E_\gamma - E_{\gamma'})^2(E_{\gamma_1} - E_{\gamma'})} \right] g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} g_1^{\gamma\gamma'} \\ &\quad - \sum_{\gamma_1, \gamma_2} e^{-iE_{\gamma'}t} \frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma_1\gamma'} \eta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma'})^2(E_{\gamma_1} - E_{\gamma'})^2(E_{\gamma_2} - E_{\gamma'})}. \end{aligned} \quad (70)$$

In the end of this subsection, we would like to point out that the main purpose introducing the g -product decomposition and calculating the contractions and anti-contractions of g -product is to eliminate the apparent divergences and find out all the limitations from the contributions of g -product contraction terms. This is important to express the results with the physical significance.

III. IMPROVED FORMS OF PERTURBED SOLUTION OF DYNAMICS

In fact, the final aim using the g -product decomposition and then calculating the limitation of the contraction terms is to absorb the partial contributions from the high order approximations of off-diagonal elements of the perturbing Hamiltonian matrix into the lower order approximations in our improved scheme of perturbation theory. In this section, making use of the skills and methods stated in previous section, we can obtain the zeroth, first, second and third order improved forms of perturbed solutions with the above features.

In mathematics, the process to obtain the improved forms of perturbed solutions is a kind of technology to deal with an infinite series, that is, according to some principles and the general term form to rearrange those terms with

the same features together forming a group, then sum all of the terms in such a particular group that they become a compact function at a given precision, finally this infinite series is transformed into a new series form that directly relates to the studied problem. More concretely speaking, since we concern the system evolution with time t , we take those terms with $(-iy_it)e^{-ix_it}$, $(-iy_it)^2e^{-ix_it}/2!$ and $(-iy_it)^3e^{-ix_it}/3!$, \dots with the same factor function f together forming a group, then sum them to obtain an exponential function $f \exp[-i(x_i + y_i)t]$. The physical reason to do this is that such an exponential function represents the system evolution in theory and it has the obvious physical significance in the calculation of transition probability and perturbed energy. Through rearranging and summing, those terms with factors $t^a e^{-iE_{\gamma_i}t}$, ($a = 1, 2, \dots$) in the higher order approximations are absorbed into the improved lower approximations, we thus can advance the precision, particularly, when the evolution time t is long enough. We can call it “dynamical rearrangement and summation” method.

A. Improved form of the zeroth order perturbed solution of dynamics

Let us start with the zeroth order perturbed solution of dynamics. In the usual perturbation theory, it is well-known

$$|\Psi^{(0)}(t)\rangle = \sum_{\gamma} e^{-iE_{\gamma}t} \langle \Phi^{\gamma} | \Psi(0) \rangle |\Phi^{\gamma}\rangle = \sum_{\gamma\gamma'} e^{-iE_{\gamma}t} \delta_{\gamma\gamma'} a_{\gamma'} |\Phi^{\gamma}\rangle, \quad (71)$$

where $a_{\gamma'} = \langle \Phi^{\gamma'} | \Psi(0) \rangle$. Now, we would like to improve it so that it can absorb the partial contributions from higher order approximations. Actually, we can find that $A_2(c)$ and $A_3(nn, c)$ have the terms proportional to $(-it)$

$$(-it)e^{-iE_{\gamma}t} \left[\sum_{\gamma_1} \frac{1}{E_{\gamma} - E_{\gamma_1}} |g_1^{\gamma\gamma_1}|^2 \right] \delta_{\gamma\gamma'}, \quad (72)$$

$$(-it)e^{-iE_{\gamma}t} \left[\sum_{\gamma_1, \gamma_2} \frac{1}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \right] \delta_{\gamma\gamma'}. \quad (73)$$

Introduce the notation

$$G_{\gamma}^{(2)} = \sum_{\gamma_1} \frac{1}{E_{\gamma} - E_{\gamma_1}} |g_1^{\gamma\gamma_1}|^2, \quad (74)$$

$$G_{\gamma}^{(3)} = \sum_{\gamma_1, \gamma_2} \frac{1}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma}. \quad (75)$$

It is clear that $G_{\gamma}^{(a)}$ has the energy dimension, and we will see that it can be called the a th revised energy. Let us add the terms (72), (73) and the related terms in $A_4(te^{-iE_{\gamma}t}, D)$, $A_4(t^2e^{-iE_{\gamma}t}, D)$, $A_5(te^{-iE_{\gamma}t}, D)$, $A_5(t^2e^{-iE_{\gamma}t}, D)$, $A_6(t^2e^{-iE_{\gamma}t}, D)$ and $A_6(t^3e^{-iE_{\gamma}t})$ given in Appendix A together, that is,

$$\begin{aligned} A_{10}^{\gamma\gamma'}(t) = & e^{-iE_{\gamma}t} \left[1 + (-it) \left(G_{\gamma}^{(2)} + G_{\gamma}^{(3)} + G_{\gamma}^{(4)} + G_{\gamma}^{(5)} \right) \right. \\ & \left. + \frac{(-it)^2}{2!} \left(G_{\gamma}^{(2)} + G_{\gamma}^{(3)} \right)^2 + \frac{(-it)^2}{2!} 2G_{\gamma}^{(2)}G_{\gamma}^{(4)} + \dots \right] \delta_{\gamma\gamma'}, \end{aligned} \quad (76)$$

Although we have not finished more calculations, from the mathematical symmetry and physical concept, we can think

$$\begin{aligned} A_{10}^{\gamma\gamma'} = & e^{-iE_{\gamma}t} \left[1 + (-it) \left(G_{\gamma}^{(2)} + G_{\gamma}^{(3)} + G_{\gamma}^{(4)} + G_{\gamma}^{(5)} \right) \right. \\ & \left. + \frac{(-it)^2}{2!} \left(G_{\gamma}^{(2)} + G_{\gamma}^{(3)} + G_{\gamma}^{(4)} + G_{\gamma}^{(5)} \right)^2 + \dots \right] \delta_{\gamma\gamma'}, \end{aligned} \quad (77)$$

New terms added to the above equation ought to, we think, appear at $A_7(t)$, $A_8(t)$, $A_9(t)$ and $A_{10}(t)$, or come from the point of view introducing the higher approximations. So we have

$$A_{10}^{\gamma\gamma'}(t) = e^{-i(E_{\gamma} + G_{\gamma}^{(2)} + G_{\gamma}^{(3)} + G_{\gamma}^{(4)} + G_{\gamma}^{(5)})t} \delta_{\gamma\gamma'} \quad (78)$$

and then obtain the improved form of the zeroth order perturbed solution of dynamics

$$|\Psi_{E_T}^{(0)}(t)\rangle_I = \sum_{\gamma\gamma'} A_{10} a_{\gamma'}(t) |\Phi^{\gamma}\rangle. \quad (79)$$

It is clear that $G_\gamma^{(2)}$ is real. In fact, $G_\gamma^{(3)}$ is also real. In order to prove it, we exchange the dummy indexes γ_1 and γ_2 and take the complex conjugate of $G_\gamma^{(3)}$, that is

$$\begin{aligned} G_\gamma^{(3)*} &= \sum_{\gamma_1, \gamma_2} \frac{1}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})} (g_1^{\gamma\gamma_2})^* (g_1^{\gamma_2\gamma_1})^* (g_1^{\gamma_1\gamma})^* \\ &= \sum_{\gamma_1, \gamma_2} \frac{1}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} \\ &= G_\gamma^{(3)}, \end{aligned} \quad (80)$$

where we have used the relations $(g_1^{\beta_1\beta_2})^* = g_1^{\beta_2\beta_1}$ for any β_1 and β_2 since H_1 is Hermit. Similar analyses can be applied to $G_\gamma^{(4)}$ and $G_\gamma^{(5)}$. These mean that $e^{-i(G_\gamma^{(2)} + G_\gamma^{(3)} + G_\gamma^{(4)} + G_\gamma^{(5)})t}$ is still an oscillatory factor.

B. Improved form of the first order perturbed solution of dynamics

Furthermore, in order to absorb the partial contributions from the approximation higher than zeroth order, we need to consider the contributions from off-diagonal elements in the higher order approximations.

Well-known usual first order perturbing part of dynamics is

$$|\Psi^{(1)}(t)\rangle = \sum_{\gamma, \gamma'} \left[\frac{e^{-iE_\gamma t}}{E_\gamma - E_{\gamma'}} - \frac{e^{-iE_{\gamma'} t}}{E_\gamma - E_{\gamma'}} \right] H_1^{\gamma\gamma'} |\Phi^\gamma\rangle = \sum_{\gamma, \gamma'} \left[\left(\frac{e^{-iE_\gamma t}}{E_\gamma - E_{\gamma'}} - \frac{e^{-iE_{\gamma'} t}}{E_\gamma - E_{\gamma'}} \right) g_1^{\gamma\gamma'} \right] |\Phi^\gamma\rangle. \quad (81)$$

It must be emphasized that H_1 is taken as only with the off-diagonal part for simplicity. That is, we have used the Hamiltonian redivision skill if the perturbing Hamiltonian matrix has the diagonal elements.

Thus, from $A_3(te^{-iE_\gamma t}, N)$ and $A_4(te^{-iE_\gamma t}, N)$, $A_4(t^2e^{-iE_\gamma t}, D)$, $A_5(t^2e^{-iE_\gamma t}, N)$, $A_6(t^2e^{-iE_\gamma t}, N)$, which are defined and calculated in the Appendix A, it follows that

$$\begin{aligned} A_{II}^{\gamma\gamma'}(t) &= \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma'})} \left[1 + (-it) \left(G_\gamma^{(2)} + G_\gamma^{(3)} + G_\gamma^{(4)} \right) + \frac{(-it)^2}{2!} \left(G_\gamma^{(2)} \right)^2 + \frac{(-it)^2}{2!} 2G_\gamma^{(2)}G_\gamma^{(3)} + \dots \right] g_1^{\gamma\gamma'} \\ &\quad - \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})} \left[1 + (-it) \left(G_{\gamma'}^{(2)} + G_{\gamma'}^{(3)} + G_{\gamma'}^{(4)} \right) + \frac{(-it)^2}{2!} \left(G_{\gamma'}^{(2)} \right)^2 + \frac{(-it)^2}{2!} 2G_{\gamma'}^{(2)}G_{\gamma'}^{(3)} + \dots \right] g_1^{\gamma\gamma'}. \end{aligned} \quad (82)$$

Therefore, by rewriting

$$A_{II}^{\gamma\gamma'}(t) = \left(\frac{e^{-i(E_\gamma + G_\gamma^{(2)} + G_\gamma^{(3)} + G_\gamma^{(4)})t}}{E_\gamma - E_{\gamma'}} - \frac{e^{-i(E_{\gamma'} + G_{\gamma'}^{(2)} + G_{\gamma'}^{(3)} + G_{\gamma'}^{(4)})t}}{E_\gamma - E_{\gamma'}} \right) g_1^{\gamma\gamma'}, \quad (83)$$

we obtain the improved form of the first order perturbed solution of dynamics

$$|\Psi^{(1)}(t)\rangle_I = \sum_{\gamma, \gamma'} A_{II}^{\gamma\gamma'}(t) a_{\gamma'} |\Phi^\gamma\rangle. \quad (84)$$

C. Improved second order- and third order perturbed solution

Likewise, it is not difficult to obtain

$$\begin{aligned} A_{12}^{\gamma\gamma'}(t) &= \sum_{\gamma_1} \left\{ - \left[\frac{e^{-i(E_\gamma + G_\gamma^{(2)} + G_\gamma^{(3)})t} - e^{-i(E_{\gamma_1} + G_{\gamma_1}^{(2)} + G_{\gamma_1}^{(3)})t}}{(E_\gamma - E_{\gamma_1})^2} \right] g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma} \delta_{\gamma\gamma'} \right. \\ &\quad + \left[\frac{e^{-i(E_\gamma + G_\gamma^{(2)} + G_\gamma^{(3)})t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma'})} - \frac{e^{-i(E_{\gamma_1} + G_{\gamma_1}^{(2)} + G_{\gamma_1}^{(3)})t}}{(E_\gamma - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma'})} \right. \\ &\quad \left. \left. + \frac{e^{-i(E_{\gamma'} + G_{\gamma'}^{(2)} + G_{\gamma'}^{(3)})t}}{(E_\gamma - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})} \right] g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'} \right\}, \end{aligned} \quad (85)$$

$$\begin{aligned}
A_{I3}^{\gamma,\gamma'}(t) = & \sum_{\gamma_1, \gamma_2} \left[-\frac{e^{-i(E_\gamma + G_\gamma^{(2)})t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})^2} - \frac{e^{-i(E_\gamma + G_\gamma^{(2)})t}}{(E_\gamma - E_{\gamma_1})^2(E_\gamma - E_{\gamma_2})} \right. \\
& + \frac{e^{-i(E_{\gamma_1} + G_{\gamma_1}^{(2)})t}}{(E_\gamma - E_{\gamma_1})^2(E_{\gamma_1} - E_{\gamma_2})} - \frac{e^{-i(E_{\gamma_2} + G_{\gamma_2}^{(2)})t}}{(E_\gamma - E_{\gamma_2})^2(E_{\gamma_1} - E_{\gamma_2})} \left. \right] g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} \delta_{\gamma\gamma'} \\
& - \sum_{\gamma_1} \left[\frac{e^{-i(E_\gamma + G_\gamma^{(2)})t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma'})^2} + \frac{e^{-i(E_\gamma + G_\gamma^{(2)})t}}{(E_\gamma - E_{\gamma_1})^2(E_\gamma - E_{\gamma'})} \right] g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'} \\
& + \sum_{\gamma_1, \gamma_2} \left[\frac{e^{-i(E_\gamma + G_\gamma^{(2)})t} \eta_{\gamma\gamma_2}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})} - \frac{e^{-i(E_{\gamma_1} + G_{\gamma_1}^{(2)})t} \eta_{\gamma_1\gamma'}}{(E_\gamma - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma'})} \right. \\
& + \left. \frac{e^{-i(E_{\gamma_2} + G_{\gamma_2}^{(2)})t} \eta_{\gamma\gamma_2}}{(E_\gamma - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_2} - E_{\gamma'})} \right] g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma'}. \tag{86}
\end{aligned}$$

Therefore, the improved forms of the second- and third order perturbed solutions are, respectively,

$$|\Psi^{(2)}(t)\rangle_I = \sum_{\gamma, \gamma'} A_{I2}^{\gamma,\gamma'}(t) a_{\gamma'} |\Phi^\gamma\rangle, \tag{87}$$

$$|\Psi^{(3)}(t)\rangle_I = \sum_{\gamma, \gamma'} A_{I3}^{\gamma,\gamma'}(t) a_{\gamma'} |\Phi^\gamma\rangle. \tag{88}$$

D. Summary

Obviously, our improved form of perturbed solution of dynamics up to the third order approximation is

$$|\Psi(t)\rangle = \sum_{i=0}^3 |\Psi^{(i)}(t)\rangle_I + \mathcal{O}(H_1^4). \tag{89}$$

However, this solution absorbs the contributions from the whole $A_l^{\gamma\gamma'}(te)$, $A_l^{\gamma\gamma'}(t^2e)$ parts up to the fifth order approximation and the whole $A_l^{\gamma\gamma'}(t^2e)$, $A_l^{\gamma\gamma'}(t^3e)$ parts in the sixth order approximation. After considering the contractions and anti-contractions, we get the result corresponds to the replacement

$$e^{-iE_{\gamma_i}t} \rightarrow e^{-i\tilde{E}_{\gamma_i}t}, \tag{90}$$

in the $A_l^{\gamma\gamma'}(e)$ part, where

$$\tilde{E}_{\gamma_i} = E_{\gamma_i} + h^{\gamma_i} + \sum_{a=2} G_{\gamma_i}^{(a)}, \tag{91}$$

$i = 0, 1, 2, \dots$, and $\gamma_0 = \gamma$. Here, we have absorbed the possible contributions from the diagonal elements of the perturbing Hamiltonian matrix. Although the upper bound of summation index a is different from the approximation order in the finished calculations, we can conjecture that it may be taken to at least 5 based on the consideration from the physical concept and mathematical symmetry. For $a \geq 5$, their forms should be similar. From our point of view, such form is so delicate that its form happens impossibly by accident. Perhaps, there is a fundamental formula within it. Nevertheless, we have no idea of how to prove it strictly and generally at present.

Actually, as soon as we carry out further calculations, we can absorb the contributions from higher order approximations. Moreover, these calculations are not difficult and are programmable because we only need to calculate the limitation and summation. Therefore, the advantages of our solution have been made clear in our improved forms of perturbed solution of dynamics. In other words, they offer clear evidences to show our improved scheme is better than the existing method in the precision and efficiency. In the following several sections, we will clearly demonstrate these problems.

IV. IMPROVED TRANSITION PROBABILITY AND REVISED FERMI'S GOLDEN RULE

One of the interesting applications of our perturbed solution is the calculation of transition probability in general quantum systems independent of time. It ameliorates the well-known conclusion because our solution absorbs the

contributions from the high order approximations of the perturbing Hamiltonian. Moreover, in terms of our improved forms of perturbed solution, it is easy to obtain the high order transition probability. In addition, for the case of sudden perturbation, our scheme is also suitable.

Let us start with the following perturbing expansion of state evolution with time t ,

$$|\Psi(t)\rangle = \sum_{\gamma} c_{\gamma}(t) |\Phi^{\gamma}\rangle = \sum_{n=0}^{\infty} \sum_{\gamma} c_{\gamma}^{(n)}(t) |\Phi^{\gamma}\rangle. \quad (92)$$

When we take the initial state as $|\Phi^{\beta}\rangle$, from our improved first order perturbed solution, we immediately obtain

$$c_{\gamma, I}^{(1)} = \frac{e^{-i\tilde{E}_{\gamma}t} - e^{-i\tilde{E}_{\beta}t}}{E_{\gamma} - E_{\beta}} g_1^{\gamma\beta}, \quad (93)$$

where

$$\tilde{E}_{\gamma_i} = E_{\gamma_i} + h_1^{\gamma_i} + G_{\gamma_i}^{(2)} + G_{\gamma_i}^{(3)} + G_{\gamma_i}^{(4)}. \quad (94)$$

Here, we use the subscript ‘‘I’’ for distinguishing it from the usual result. Omitting a unimportant phase factor $e^{-i\tilde{E}_{\gamma}t}$, we can rewrite it as

$$c_{\gamma, I}^{(1)} = \frac{g_1^{\gamma\beta}}{E_{\gamma} - E_{\beta}} \left(1 - e^{i\tilde{\omega}_{\gamma\beta}t}\right), \quad (95)$$

where $\tilde{\omega}_{\gamma\beta} = \tilde{E}_{\gamma} - \tilde{E}_{\beta}$. Obviously it is different from the well known conclusion

$$c_{\gamma}^{(1)} = \frac{g_1^{\gamma\beta}}{E_{\gamma} - E_{\beta}} \left(1 - e^{i\omega_{\gamma\beta}t}\right), \quad (96)$$

where $\omega_{\gamma\beta} = E_{\gamma} - E_{\beta}$. Therefore, our result contains the partial contributions from the high order approximations.

Considering the transition probability from $|\Phi^{\beta}\rangle$ to $|\Phi^{\gamma}\rangle$ after time T , we have

$$P_I^{\gamma\beta}(t) = \frac{|g_1^{\gamma\beta}|^2}{\omega_{\gamma\beta}^2} \left|1 - e^{i\tilde{\omega}_{\gamma\beta}T}\right|^2 = |g_1^{\gamma\beta}|^2 \frac{\sin^2(\tilde{\omega}_{\gamma\beta}T/2)}{(\omega_{\gamma\beta}/2)^2}. \quad (97)$$

In terms of the relation

$$\sin^2 x - \sin^2 y = \frac{1}{2} [\cos(2y) - \cos(2x)], \quad (98)$$

we have the revision part of transition probability

$$\Delta P_I^{\gamma\beta}(t) = 2 |g_1^{\gamma\beta}|^2 \frac{\cos(\omega_{\gamma\beta}T) - \cos(\tilde{\omega}_{\gamma\beta}T)}{(\omega_{\gamma\beta})^2}. \quad (99)$$

If plotting

$$\frac{\sin^2(\tilde{\omega}_{\gamma\beta}T/2)}{(\omega_{\gamma\beta}/2)^2} = \left(\frac{\tilde{\omega}_{\gamma\beta}}{\omega_{\gamma\beta}}\right)^2 \frac{\sin^2(\tilde{\omega}_{\gamma\beta}T/2)}{(\tilde{\omega}_{\gamma\beta}/2)^2}, \quad (100)$$

we can see that it has a well-defined peak centered at $\tilde{\omega}_{\gamma\beta} = 0$. Just as what has been done in the usual case, we can extend the integral range as $-\infty \rightarrow \infty$. Thus, the revised Fermi's golden rule

$$w = w_F + \Delta w, \quad (101)$$

where the usual Fermi's golden rule is [8]

$$w_F = 2\pi\rho(E_{\beta}) |g_1^{\gamma\beta}|^2, \quad (102)$$

in which w means the transition velocity, $\rho(E_\gamma)$ is the density of final state and we have used the integral formula

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} = \pi. \quad (103)$$

while the revision part is

$$\Delta w = 2 \int_{-\infty}^{\infty} dE_\gamma \rho(E_\gamma) \left| g_1^{\gamma\beta} \right|^2 \frac{\cos(\omega_{\gamma\beta} T) - \cos(\tilde{\omega}_{\gamma\beta} T)}{T (\omega_{\gamma\beta})^2}. \quad (104)$$

It is clear that $\tilde{\omega}_{\gamma\beta}$ is a function of E_γ , and then a function of $\omega_{\gamma\beta}$. For simplicity, we only consider $\tilde{\omega}_{\gamma\beta}$ to its second order approximation, that is

$$\tilde{\omega}_{\gamma\beta} = \tilde{\omega}(\omega_{\gamma\beta}) = \omega_{\gamma\beta} + \sum_{\gamma_1} \left[\frac{|g_1^{\gamma\gamma_1}|^2}{\omega_{\gamma\beta} - \omega_{\gamma_1\beta}} - \frac{|g_1^{\beta\gamma_1}|^2}{\omega_{\beta\gamma_1}} \right] + \mathcal{O}(H_1^3). \quad (105)$$

Again based on $dE_\gamma = d\omega_{\gamma\beta}$, we have

$$\Delta w = 2 \int_{-\infty}^{\infty} d\omega_{\gamma\beta} \rho(\omega_{\gamma\beta} + E_\beta) \left| g_1^{\gamma\beta} \right|^2 \frac{\cos[\omega_{\gamma\beta} T] - \cos[\tilde{\omega}_{\gamma\beta}(\omega) T]}{T (\omega_{\gamma\beta})^2}. \quad (106)$$

It seems to not to be easy to deduce the general form of this integral. In order to simplify it, we can use the fact that $\tilde{\omega}_{\gamma\beta} - \omega_{\gamma\beta}$ is a smaller quantity since

$$\Delta\omega_{\gamma\beta} = \tilde{\omega}_{\gamma\beta} - \omega_{\gamma\beta} = \sum_{i=2}^4 \left(G_\gamma^{(i)} - G_\beta^{(i)} \right). \quad (107)$$

For example, we can approximatively take

$$\cos(\omega_{\gamma\beta} T) - \cos(\tilde{\omega}_{\gamma\beta} T) \approx T (\tilde{\omega}_{\gamma\beta} - \omega_{\gamma\beta}) \sin(\tilde{\omega}_{\gamma\beta} T - \omega_{\gamma\beta} T), \quad (108)$$

then calculate the integral. We will study it in our other manuscript (in preparing).

Obviously, the revision comes from the contributions of high order approximations. The physical effect resulted from our solution, whether is important or unimportant, should be investigated in some concrete quantum systems. Recently, we reconsider the transition probability and perturbed energy for a Hydrogen atom in a constant magnetic field [9]. We find the results obtained by using our improved scheme are indeed more satisfying in the calculation precision and efficiency. We will discuss more examples in our future manuscripts (in preparing).

It is clear that the relevant results can be obtained from the usual results via replacing $\omega_{\gamma\beta}$ in the exponential power by using $\tilde{\omega}_{\gamma\beta}$. Thus, one thing is true — with the time t evolving, $e^{\pm i(\tilde{\omega}_{\gamma\beta} t/2)}$ term in the improved transition probability can be very different from $e^{\pm i(\omega_{\gamma\beta} t/2)}$ in the traditional one, which might lead to totally different results. To save the space, we do not intend to discuss more here.

In fact, there is no any difficulty to obtain the second- and three order transition probability in terms of our improved forms of perturbed solution in the previous section. More higher order transition probability can be given effectively and accurately by our scheme.

V. IMPROVED FORMS OF PERTURBED ENERGY AND PERTURBED STATE

Now we study how to calculate the improved forms of perturbed energy and perturbed state. For simplicity, we only study them concerning the improved second order approximation. Based on the experience from the skill one in Sec. VI, we can, in fact, set a new \tilde{E} and then use the technology in the usual perturbative theory. That is, we denote

$$\tilde{E}_{\gamma_i} = E_{\gamma_i} + G_{\gamma_i}^{(2)} + G_{\gamma_i}^{(3)}. \quad (109)$$

$$\begin{aligned}
|\Psi(t)\rangle = & \sum_{\gamma, \gamma'} \left\{ e^{-i\tilde{E}_{\gamma}t} \delta_{\gamma\gamma'} + \left[\frac{e^{-i\tilde{E}_{\gamma}t} - e^{-i\tilde{E}_{\gamma'}t}}{E_{\gamma} - E_{\gamma'}} \right] g_1^{\gamma\gamma'} - \sum_{\gamma_1} \frac{e^{-i\tilde{E}_{\gamma}t} - e^{-i\tilde{E}_{\gamma_1}t}}{(E_{\gamma} - E_{\gamma_1})^2} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma} \delta_{\gamma\gamma'} \right. \\
& + \sum_{\gamma_1} \left[\frac{e^{-i\tilde{E}_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma'})} - \frac{e^{-i\tilde{E}_{\gamma_1}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma'})} \right. \\
& \left. \left. + \frac{e^{-i\tilde{E}_{\gamma'}t}}{(E_{\gamma} - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})} \right] g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma} \eta_{\gamma\gamma'} \right\} a_{\gamma'} |\Phi^{\gamma}\rangle + \mathcal{O}(H_1^3). \tag{110}
\end{aligned}$$

Note that E_{γ_i} can contain the diagonal element $h_1^{\gamma_i}$ of the original H_1 , and we do not obviously write $h_1^{\gamma_i}$ and take new H_1 matrix as off-diagonal in the H_0 representation.

Because that

$$|\Psi(t)\rangle = \sum_{\gamma, \gamma'} e^{-iE_T t} \delta_{\gamma\gamma'} a_{\gamma'} |\Phi^{\gamma}\rangle, \tag{111}$$

we have

$$\begin{aligned}
E_T a_{\gamma} = & \tilde{E}_{\gamma} a_{\gamma} + \sum_{\gamma'} \left\{ \frac{\tilde{E}_{\gamma} - \tilde{E}_{\gamma'}}{E_{\gamma} - E_{\gamma'}} g_1^{\gamma\gamma'} - \sum_{\gamma_1} \frac{\tilde{E}_{\gamma} - \tilde{E}_{\gamma_1}}{(E_{\gamma} - E_{\gamma_1})^2} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma} \delta_{\gamma\gamma'} + \sum_{\gamma_1} \left[\frac{\tilde{E}_{\gamma}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma'})} \right. \right. \\
& \left. \left. - \frac{\tilde{E}_{\gamma_1}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma'})} + \frac{\tilde{E}_{\gamma'}}{(E_{\gamma} - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})} \right] g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma} \eta_{\gamma\gamma'} \right\} a_{\gamma'}. \tag{112}
\end{aligned}$$

In the usual perturbation theory, H_1 is taken as a perturbing part with the form

$$H_1 = \lambda v, \tag{113}$$

where λ is a real number that is called the perturbing parameter. It must be emphasized that \tilde{E}_{γ_i} can be taken as explicitly independent perturbing parameter λ , because we introduce λ as a formal multiplier after redefinition. In other words, \tilde{E}_{γ_i} has absorbed those terms adding to it and formed a new quantity. This way has been seen in our Hamiltonian redivision skill. Without loss of generality, we further take H_1 only with the off-diagonal form, that is

$$H_1^{\gamma_1\gamma_2} = g_1^{\gamma_1\gamma_2} = \lambda v^{\gamma_1\gamma_2}. \tag{114}$$

Then, we expand both the desired expansion coefficients a_{γ} and the energy eigenvalues E_T in a power series of perturbation parameter λ :

$$E_T = \sum_{l=0}^{\infty} \lambda^l E_{T,I}^{(l)}, \quad a_{\gamma} = \sum_{l=0}^{\infty} \lambda^l a_{\gamma,I}^{(l)}. \tag{115}$$

1. Improved 0th approximation

If we set $\lambda = 0$, eq.(112) yields

$$E_{T,I}^{(0)} a_{\gamma,I}^{(0)} = \tilde{E}_{\gamma} a_{\gamma,I}^{(0)}, \tag{116}$$

where γ runs over all levels. Actually, let us focus on the level $\gamma = \beta$, then

$$E_{T,I}^{(0)} = \tilde{E}_{\beta}. \tag{117}$$

When the initial state is taken as $|\Phi^{\beta}\rangle$,

$$a_{\gamma,I}^{(0)} = \delta_{\gamma\beta}. \tag{118}$$

Obviously, the improved form of perturbed energy is different from the results in the usual perturbative theory because it absorbs the contributions from the higher order approximations. However, the so-call improved form of perturbed state is the same as the usual result.

2. Improved 1st approximation

Again from eq.(112) it follows that

$$E_{T,I}^{(0)} a_{\gamma;I}^{(1)} + E_{T,I}^{(1)} a_{\gamma;I}^{(0)} = \tilde{E}_\gamma a_{\gamma;I}^{(1)} + \sum_{\gamma'} \frac{\tilde{E}_\gamma - \tilde{E}_{\gamma'}}{E_\gamma - E_{\gamma'}} v^{\gamma\gamma'} a_{\gamma';I}^{(0)}. \quad (119)$$

When $\gamma = \beta$, it is easy to obtain

$$E_{T,I}^{(1)} = 0. \quad (120)$$

If $\gamma \neq \beta$, then

$$a_{\gamma;I}^{(1)} = -\frac{1}{(E_\gamma - E_\beta)} v^{\gamma\beta}. \quad (121)$$

It is clear that the first order results are the same as the one in the usual perturbative theory.

3. Improved 2nd approximation

Likewise, the following equation

$$\begin{aligned} E_{T,I}^{(2)} a_{\gamma;I}^{(0)} + E_{T,I}^{(1)} a_{\gamma;I}^{(1)} + E_{T,I}^{(0)} a_{\gamma;I}^{(2)} &= \tilde{E}_\gamma a_{\gamma;I}^{(2)} + \sum_{\gamma'} \frac{\tilde{E}_\gamma - \tilde{E}_{\gamma'}}{E_\gamma - E_{\gamma'}} v^{\gamma\gamma'} a_{\gamma';I}^{(1)} - \sum_{\gamma_1, \gamma'} \frac{\tilde{E}_\gamma - \tilde{E}_{\gamma_1}}{(E_\gamma - E_{\gamma_1})^2} v^{\gamma\gamma_1} v^{\gamma_1\gamma} \delta_{\gamma\gamma'} a_{\gamma';I}^{(0)} \\ &+ \sum_{\gamma_1, \gamma'} \left[\frac{\tilde{E}_\gamma}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma'})} - \frac{\tilde{E}_{\gamma_1}}{(E_\gamma - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma'})} + \frac{\tilde{E}_{\gamma'}}{(E_\gamma - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})} \right] v^{\gamma\gamma_1} v^{\gamma_1\gamma} \eta_{\gamma\gamma'} a_{\gamma';I}^{(0)}. \end{aligned} \quad (122)$$

is obtained and it yields

$$E_{T,I}^{(2)} = 0, \quad (123)$$

if we take $\gamma = \beta$. When $\gamma \neq \beta$, we have

$$a_{\gamma;I}^{(2)} = \sum_{\gamma_1} \frac{1}{(E_\gamma - E_\beta)(E_{\gamma_1} - E_\beta)} v^{\gamma\gamma_1} v^{\gamma_1\beta} \eta_{\gamma\beta}. \quad (124)$$

It is consistent with the off-diagonal part of usual result. In fact, since we have taken $H_1^{\gamma\gamma'}$ to be off-diagonal, it does not have a diagonal part. However, we think its form is more appropriate. In addition, we do not consider the revision part introduced by normalization. While $E_{T,I}^{(2)} = 0$ is a new result.

4. Summary

Now we can see, up to the improved second order approximation:

$$E_{T,\beta} \approx \tilde{E}_\beta = E_\beta + G_\beta^{(2)} + G_\beta^{(3)}. \quad (125)$$

Compared with the usual one, they are consistent at the former two orders. It is not strange since the physical law is the same. However, our improved form of perturbed energy contains a third order term. In other words, our solution might be effective in order to obtain the contribution from high order approximations. The possible physical reason is that a redefined form of the solution is obtained.

In special, when we allow the $H_1^{\gamma\gamma'}$ to have the diagonal elements, the improved second order approximation becomes

$$E_{T,\beta} \approx E_\beta + h_1^\beta + G_\beta^{(2)} + G_\beta^{(3)}. \quad (126)$$

Likewise, if we redefine

$$\tilde{E}_{\gamma_i} = E_{\gamma_i} + h_1^{\gamma_i} + G_{\gamma_i}^{(2)} + G_{\gamma_i}^{(3)} + G_{\gamma_i}^{(4)}. \quad (127)$$

Thus, only considering the first order approximation, we can obtain

$$E_{T,\beta} \approx E_\beta + h_1^\beta + G_\beta^{(2)} + G_\beta^{(3)} + G_\beta^{(4)}. \quad (128)$$

In fact, the reason is our conjecture in the previous section. The correct form of redefined \tilde{E}_{γ_i} should be

$$E_{T,\beta} \approx E_\beta + h_1^\beta + G_\beta^{(2)} + G_\beta^{(3)} + G_\beta^{(4)} + G_\beta^{(5)} + \dots. \quad (129)$$

This implies that our improved scheme absorbs the partial even whole significant contributions from the high order approximations. In addition, based on the fact that the improved second approximation is actually zero, it is possible that this implies our solution will fade down more rapidly than the solution in the usual perturbative theory.

Actually, the main advantage of our solution is in dynamical development. The contributions from the high order approximations play more important roles in the relevant physical problems such as the entanglement dynamics and decoherence process. For the improved perturbed energy, its high order part has obvious physical meaning. But, for the improved form of perturbed state, we find them to be the same as the existed perturbation theory up to the second approximation.

VI. EXAMPLE AND APPLICATION

In order to concretely illustrate that our exact solution and the improved scheme of perturbation theory are indeed more effective and more accurate, let us study a simple example: two-state system, which appears in the most of quantum mechanics textbooks. Its Hamiltonian can be written as

$$H = \begin{pmatrix} E_1 & V_{12} \\ V_{21} & E_2 \end{pmatrix}, \quad (130)$$

where we have used the the basis formed by the unperturbed energy eigenvectors, that is

$$|\Phi^1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\Phi^2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (131)$$

In other words:

$$H_0|\Phi^\gamma\rangle = E_\gamma|\Phi^\gamma\rangle, \quad (\gamma = 1, 2) \quad (132)$$

where

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}. \quad (133)$$

Thus, this means the perturbing Hamiltonian is taken as

$$H_1 = \begin{pmatrix} 0 & V_{12} \\ V_{21} & 0 \end{pmatrix}. \quad (134)$$

This two state system has the following eigen equation

$$H|\Psi^\gamma\rangle = E_\gamma^T|\Psi^\gamma\rangle. \quad (135)$$

It is easy to obtain its solution: corresponding eigenvectors and eigenvalues

$$|\Psi^1\rangle = \frac{1}{\sqrt{4|V|^2 + (\omega_{21} + \omega_{21}^T)^2}} \begin{pmatrix} \omega_{21} + \omega_{21}^T \\ -2V_{21} \end{pmatrix}, \quad (136)$$

$$|\Psi^2\rangle = \frac{1}{\sqrt{4|V|^2 + (\omega_{21} - \omega_{21}^T)^2}} \begin{pmatrix} \omega_{21} - \omega_{21}^T \\ -2V_{21} \end{pmatrix}; \quad (137)$$

$$E_1^T = \frac{1}{2} (E_1 + E_2 - \omega_{21}^T), \quad E_2^T = \frac{1}{2} (E_1 + E_2 + \omega_{21}^T); \quad (138)$$

where $|V| = |V_{12}| = |V_{21}|$, $\omega_{21} = E_2 - E_1$, $\omega_{21}^T = E_2^T - E_1^T = \sqrt{4|V|^2 + \omega_{21}^2}$, and we have set $E_2 > E_1$ without loss of generality.

Obviously the transition probability from state 1 to state 2 is

$$P^T(1 \rightarrow 2) = |\langle \Phi^2 | e^{-iHt} | \Phi^1 \rangle|^2 = \left| \sum_{\gamma_1, \gamma_2=1}^2 \langle \Phi^2 | \Psi^{\gamma_1} \rangle \langle \Psi^{\gamma_1} | e^{-iHt} | \Psi^{\gamma_2} \rangle \langle \Psi^{\gamma_2} | \Phi^1 \rangle \right|^2 = |V|^2 \frac{\sin^2(\omega_{21}^T t/2)}{(\omega_{21}^T/2)^2}. \quad (139)$$

In the usual perturbation theory, up to the second order approximation, the well-known perturbed energies are

$$E_1^P = E_1 - \frac{|V|^2}{\omega_{21}}, \quad E_2^P = E_1 + \frac{|V|^2}{\omega_{21}}. \quad (140)$$

While, under the first order approximation, the transition probability from state 1 to state 2 is

$$P(1 \rightarrow 2) = |V|^2 \frac{\sin^2(\omega_{21}t/2)}{(\omega_{21}/2)^2}. \quad (141)$$

Using our improved scheme, only to the first order approximation, we get the corresponding perturbed energies

$$\tilde{E}_1 = E_1 - \frac{|V|^2}{\omega_{21}} + \frac{|V|^4}{\omega_{21}^3}, \quad \tilde{E}_2 = E_1 + \frac{|V|^2}{\omega_{21}} - \frac{|V|^4}{\omega_{21}^3}, \quad (142)$$

where we have used the facts that

$$G_1^{(2)} = -\frac{|V|^2}{\omega_{21}} = -G_2^{(2)}, \quad G_1^{(3)} = G_2^{(3)} = 0, \quad G_1^{(4)} = \frac{|V|^4}{\omega_{21}^3} = -G_2^{(4)}. \quad (143)$$

Obviously, under the first order approximation, our scheme yields the transition probability from state 1 to state 2 as

$$P_1(1 \rightarrow 2) = |V|^2 \frac{\sin^2(\tilde{\omega}_{21}t/2)}{(\omega_{21}/2)^2}. \quad (144)$$

where $\tilde{\omega}_{21} = \tilde{E}_2 - \tilde{E}_1$. Therefore we can say our scheme is more effective. Moreover, we notice that

$$E_1^T = E_1 - \frac{|V|^2}{\omega_{21}} + \frac{|V|^4}{\omega_{21}^3} + \mathcal{O}(|V|^6) = \tilde{E}_1 + \mathcal{O}(|V|^6) = E_1^P + \frac{|V|^4}{\omega_{21}^3} + \mathcal{O}(|V|^6), \quad (145)$$

$$\tilde{E}_2^T = E_1 + \frac{|V|^2}{\omega_{21}} - \frac{|V|^4}{\omega_{21}^3} + \mathcal{O}(|V|^6) = \tilde{E}_2 + \mathcal{O}(|V|^6) = E_2^P - \frac{|V|^4}{\omega_{21}^3} + \mathcal{O}(|V|^6). \quad (146)$$

and

$$P^T(1 \rightarrow 2) = |V|^2 \frac{\sin^2(\omega_{21}t/2)}{(\omega_{21}/2)^2} + |V|^2 \left[\frac{\sin(\omega_{21}t)}{2(\omega_{21}/2)^2} - \frac{\sin^2(\omega_{21}t/2)}{(\omega_{21}/2)^3} \right] (\tilde{\omega}_{21} - \omega_{21}) + \mathcal{O}[(\tilde{\omega}_{21} - \omega_{21})^2] \quad (147)$$

$$= P_1(1 \rightarrow 2) - |V|^2 \frac{\sin^2(\omega_{21}t/2)}{(\omega_{21}/2)^3} (\tilde{\omega}_{21} - \omega_{21}) + \mathcal{O}[(\tilde{\omega}_{21} - \omega_{21})^2] \quad (148)$$

$$= P(1 \rightarrow 2) + |V|^2 \left[\frac{\sin(\omega_{21}t)}{2(\omega_{21}/2)^2} - \frac{\sin^2(\omega_{21}t/2)}{(\omega_{21}/2)^3} \right] (\tilde{\omega}_{21} - \omega_{21}) + \mathcal{O}[(\tilde{\omega}_{21} - \omega_{21})^2]. \quad (149)$$

Therefore, we can say that our scheme is more accurate.

VII. DISCUSSION AND CONCLUSION

In this paper, our improved scheme of perturbation theory is proposed based on our exact solution in general quantum systems [3]. Because our exact solution has a general term that is a c -number function and proportional to power of the perturbing Hamiltonian, this provides the probability considering the partial contributions from the high order even all order approximations. While our dynamical rearrangement and summation method helps us to

realize this probability. Just as the contributions from the high order even all order approximations are absorbed to the lower order approximations, our scheme becomes an improved one.

It must be emphasized that our improved scheme of perturbation theory is proposed largely dependent on the facts that we find and develop a series of skills and methods. From that the Hamiltonian redivision skill overcomes the flaw in the usual perturbation theory, improves the calculation precision, extends the applicable range and removes the possible degeneracies to that the perturbing Hamiltonian matrix product decomposition method separates the contraction terms and anti-contraction terms, eliminates the apparent divergences in the power series of the perturbing Hamiltonian and provides the groundwork of “dynamical rearrangement and summation”, we have seen these ideas, skills and methods to be very useful.

Actually, the start point that delays to introduce the perturbing parameter as possible plays an important even key role in our improved scheme of perturbation theory. It enlightens us to seek for the above skills and methods.

From our exact solution transferring to our improved scheme of perturbation theory we does not directly use the cut-off approximation, but first deals with the power series of perturbation so that the contributions from some high order even all order approximations can be absorbed into the lower orders than the cut-off order as possible. Hence, our improved scheme of perturbation theory is physically reasonable, mathematically clear and methodologically skillful. This provides the guarantee achieving high efficiency and high precision. Through finding the improved forms of perturbed solutions of dynamics, we generally demonstrate this conclusion. Furthermore, we prove the correctness of this conclusion via calculating the improved form of transition probability, perturbed energy and perturbed state. Specially, we obtain the revised Fermi’s golden rule. Moreover, we illustrate the advantages of our improved scheme in an easy understanding example of two-state system. All of this implies the physical reasons and evidences why our improved scheme of perturbation theory is actually calculable, operationally efficient, conclusively more accurate.

From the features of our improved scheme, we believe that it will have interesting applications in the calculation of entanglement dynamics and decoherence process as well as the other physical quantities dependent on the expanding coefficients.

In fact, a given lower order approximation of improved form of the perturbed solution absorbing the partial contributions from the higher order even all order approximations is obtained by our dynamical rearrangement and summation method, just like “Fynmann figures” summation” that has been done in the quantum field theory. It is emphasized that these contributions have to be significant in physics. Considering time evolution form is our physical ideas and absorbing the high order approximations with the factors $t^a e^{-iE_{\gamma_i} t}$, ($a = 1, 2, \dots$) to the improved lower order approximations definitely can advance the precision. Therefore, using our dynamical rearrangement and summation method is appropriate and reasonable in our view.

For a concrete example, except for some technological and calculational works, it needs the extensive physical background knowledge to account for the significance of related results. That is, since the differences of the related conclusions between our improved scheme and the usual perturbation theory are in high order approximation parts, we have to study the revisions (differences) to find out whether they are important or unimportant to the studied problems. In addition, our conjecture about the perturbed energy is based on physical symmetry and mathematical consideration, it is still open at the strict sense. As to the degenerate cases including specially, vanishing the off-diagonal element of the perturbing Hamiltonian matrix between any two degenerate levels, we have discussed how to deal with them, except for the very complicated cases that the degeneracy can not be completely removed by the diagonalization of the degenerate subspaces trick and the Hamiltonian redivision skill as well as the off-diagonal element of the perturbing Hamiltonian matrix between any two degenerate levels are not vanishing when the remained degeneracies are allowed.

It must be emphasized that the study on the time evolution operator plays a central role in quantum dynamics and perturbation theory. Because of the universal significance of our general and explicit expression of the time evolution operator, we wish that it will have more applications in quantum theory. Besides the above studies through the perturbative method, it is more interesting to apply our exact solution to the formalization study of quantum dynamics in order to further and more powerfully show the advantages of our exact solution.

In summary, our results can be thought of as theoretical developments of perturbation theory, and they are helpful for understanding the theory of quantum mechanics and providing some powerful tools for the calculation of physical quantities in general quantum systems. Together with our exact solution [3] and open system dynamics [6], they can finally form the foundation of theoretical formulism of quantum mechanics in general quantum systems. Further study on quantum mechanics of general quantum systems is on progressing.

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APPENDIX A: THE CALCULATIONS OF THE HIGH ORDER TERMS

Since we have taken the H_1 only with the off-diagonal part, it is enough to calculate the contributions from them. In Sec. VI the contributions from the first, second and third order approximations have been given. In this appendix, we would like to find the contributions from the fourth to the sixth order approximations. The calculational technologies used by us are mainly to the limit process, dummy index changing and summation, as well as the replacement $g_1^{\gamma_i \gamma_j} \eta_{\gamma_i \gamma_j} = g_1^{\gamma_i \gamma_j}$ since $g_1^{\gamma_i \gamma_j}$ has been off-diagonal. These calculations are not difficult, but are a little lengthy.

1. $l = 4$ case

For the fourth order approximation, its contributions from the first decompositions consists of eight terms:

$$\begin{aligned} A_4^{\gamma\gamma'} = & A_4^{\gamma\gamma'}(ccc) + A_4^{\gamma\gamma'}(ccn) + A_4^{\gamma\gamma'}(cnc) + A_4^{\gamma\gamma'}(ncc) \\ & + A_4^{\gamma\gamma'}(cnn) + A_4^{\gamma\gamma'}(ncn) + A_4^{\gamma\gamma'}(nnc) + A_4^{\gamma\gamma'}(nnn). \end{aligned} \quad (A1)$$

Its the former four terms have no the nontrivial second contractions, and its the fifth and seven terms have one nontrivial second contraction as follows

$$A_4^{\gamma\gamma'}(cnn) = A_4^{\gamma\gamma'}(cnn, kc) + A_4^{\gamma\gamma'}(cnn, kn), \quad (A2)$$

$$A_4^{\gamma\gamma'}(ncn) = A_4^{\gamma\gamma'}(ncn, c) + A_4^{\gamma\gamma'}(ncn, n), \quad (A3)$$

$$A_4^{\gamma\gamma'}(nnc) = A_4^{\gamma\gamma'}(nnc, ck) + A_4^{\gamma\gamma'}(nnc, nk). \quad (A4)$$

In addition, the last term in eq.(A1) has two nontrivial second contractions, and its fourth term has also the third contraction. Hence

$$A_4^{\gamma\gamma'}(nnn) = A_4^{\gamma\gamma'}(nnn, cc) + A_4^{\gamma\gamma'}(nnn, cn)A_4^{\gamma\gamma'}(nnn, nc) + A_4^{\gamma\gamma'}(nnn, nn), \quad (A5)$$

$$A_4^{\gamma\gamma'}(nnn, nn) = A_4^{\gamma\gamma'}(nnn, nn, c) + A_4^{\gamma\gamma'}(nnn, nn, n). \quad (A6)$$

All together, we have the fifteen terms that are the contributions from whole contractions and anti-contractions of the fourth order approximation.

First, let us calculate the former four terms only with the first contractions and anti-contractions, that is, with more than two δ functions (or less than two η functions)

$$\begin{aligned} A_4^{\gamma\gamma'}(ccc) = & \sum_{\gamma_1, \dots, \gamma_5} \left[\sum_{i=1}^5 (-1)^{i-1} \frac{e^{-iE_{\gamma_i} t}}{d_i(E[\gamma, 4])} \right] \left[\prod_{j=1}^4 g_1^{\gamma_j \gamma_{j+1}} \right] \left(\prod_{k=1}^3 \delta_{\gamma_k \gamma_{k+2}} \right) \delta_{\gamma_1 \gamma} \delta_{\gamma_5 \gamma'} \\ = & \sum_{\gamma_1} \left[\frac{3e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^4} - \frac{3e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})^4} - (-it) \frac{2e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^3} \right. \\ & \left. - (-it) \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})^3} + \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^2} \right] |g_1^{\gamma \gamma_1}|^4 \delta_{\gamma \gamma'}. \end{aligned} \quad (A7)$$

$$\begin{aligned} A_4^{\gamma\gamma'}(ccn) = & \sum_{\gamma_1, \dots, \gamma_5} \left[\sum_{i=1}^5 (-1)^{i-1} \frac{e^{-iE_{\gamma_i} t}}{d_i(E[\gamma, 4])} \right] \left[\prod_{j=1}^4 g_1^{\gamma_j \gamma_{j+1}} \right] \delta_{\gamma_1 \gamma_3} \delta_{\gamma_2 \gamma_4} \eta_{\gamma_3 \gamma_5} \delta_{\gamma_1 \gamma} \delta_{\gamma_5 \gamma'} \\ = & \sum_{\gamma_1} \left[-\frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma'})^2} - \frac{2e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^3 (E_{\gamma} - E_{\gamma'})} - \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma'})^2} \right. \\ & + \frac{2e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})^3 (E_{\gamma_1} - E_{\gamma'})} + \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})^2} + (-it) \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma'})} \\ & \left. + (-it) \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma'})} \right] |g_1^{\gamma \gamma_1}|^2 g_1^{\gamma \gamma_1} g_1^{\gamma_1 \gamma'} \eta_{\gamma \gamma'}. \end{aligned} \quad (A8)$$

$$\begin{aligned}
A_4^{\gamma\gamma'}(cnc) &= \sum_{\gamma_1, \dots, \gamma_5} \left[\sum_{i=1}^5 (-1)^{i-1} \frac{e^{-iE_{\gamma_i} t}}{d_i(E[\gamma, 4])} \right] \left[\prod_{j=1}^4 g_1^{\gamma_j \gamma_{j+1}} \right] \delta_{\gamma_1 \gamma_3} \eta_{\gamma_2 \gamma_4} \delta_{\gamma_3 \gamma_5} \delta_{\gamma_1 \gamma} \delta_{\gamma_5 \gamma'} \\
&= \sum_{\gamma_1, \gamma_2} \left[\frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})^3} + \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma_2})^2} + \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^3 (E_{\gamma} - E_{\gamma_2})} \right. \\
&\quad - \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})^3 (E_{\gamma_1} - E_{\gamma_2})} + \frac{e^{-iE_{\gamma_2} t}}{(E_{\gamma} - E_{\gamma_2})^3 (E_{\gamma_1} - E_{\gamma_2})} - (-it) \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})^2} \\
&\quad \left. - (-it) \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma_2})} + \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})} \right] \\
&\quad \times |g_1^{\gamma \gamma_1}|^2 |g_1^{\gamma \gamma_2}|^2 \eta_{\gamma_1 \gamma_2} \delta_{\gamma \gamma'}.
\end{aligned} \tag{A9}$$

$$\begin{aligned}
A_4^{\gamma\gamma'}(ncc) &= \sum_{\gamma_1, \dots, \gamma_5} \left[\sum_{i=1}^5 (-1)^{i-1} \frac{e^{-iE_{\gamma_i} t}}{d_i(E[\gamma, 4])} \right] \left[\prod_{j=1}^4 g_1^{\gamma_j \gamma_{j+1}} \right] \eta_{\gamma_1 \gamma_3} \delta_{\gamma_2 \gamma_4} \delta_{\gamma_3 \gamma_5} \delta_{\gamma_1 \gamma} \delta_{\gamma_5 \gamma'} \\
&= \sum_{\gamma_1} \left[\frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma'})^2} + \frac{2e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma'})^3} - \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma'})^2} \right. \\
&\quad - \frac{2e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'})^3} - \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})^2} - (-it) \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma'})^2} \\
&\quad \left. - (-it) \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'})^2} \right] |g_1^{\gamma \gamma'}|^2 g_1^{\gamma \gamma_1} g_1^{\gamma_1 \gamma'} \eta_{\gamma \gamma'}.
\end{aligned} \tag{A10}$$

Then, we calculate the three terms with the single first contraction, that is, with one δ function. Because one δ function can not eliminate the whole apparent singularity, we also need to find out the nontrivial second contraction-and/or anti-contraction terms.

$$\begin{aligned}
A_4^{\gamma\gamma'}(cnn, kc) &= \sum_{\gamma_1, \dots, \gamma_5} \left[\sum_{i=1}^5 (-1)^{i-1} \frac{e^{-iE_{\gamma_i} t}}{d_i(E[\gamma, 4])} \right] \left[\prod_{j=1}^4 g_1^{\gamma_j \gamma_{j+1}} \right] \delta_{\gamma_1 \gamma_3} \eta_{\gamma_2 \gamma_4} \eta_{\gamma_3 \gamma_5} \delta_{\gamma_2 \gamma'} \delta_{\gamma_1 \gamma} \delta_{\gamma_5 \gamma'} \\
&= \sum_{\gamma_1} \left[-\frac{2e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma'})^3} - \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma'})^2} + \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma'})^2} \right. \\
&\quad - \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})^2} - \frac{2e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'})^3 (E_{\gamma_1} - E_{\gamma'})} + (-it) \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma'})^2} \\
&\quad \left. - (-it) \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})} \right] |g_1^{\gamma \gamma'}|^2 g_1^{\gamma \gamma_1} g_1^{\gamma_1 \gamma'}.
\end{aligned} \tag{A11}$$

$$\begin{aligned}
A_4^{\gamma\gamma'}(cnn, kn) &= \sum_{\gamma_1, \dots, \gamma_5} \left[\sum_{i=1}^5 (-1)^{i-1} \frac{e^{-iE_{\gamma_i} t}}{d_i(E[\gamma, 4])} \right] \left[\prod_{j=1}^4 g_1^{\gamma_j \gamma_{j+1}} \right] \delta_{\gamma_1 \gamma_3} \eta_{\gamma_2 \gamma_4} \eta_{\gamma_3 \gamma_5} \eta_{\gamma_2 \gamma'} \delta_{\gamma_1 \gamma} \delta_{\gamma_5 \gamma'} \\
&= \sum_{\gamma_1, \gamma_2} \left[-\frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma'})^2} - \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})^2 (E_{\gamma} - E_{\gamma'})} \right. \\
&\quad - \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2})(E_{\gamma} - E_{\gamma'})} + \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma'})} \\
&\quad - \frac{e^{-iE_{\gamma_2} t}}{(E_{\gamma} - E_{\gamma_2})^2 (E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_2} - E_{\gamma'})} + \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})} \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma'})} \right] |g_1^{\gamma \gamma_1}|^2 |g_1^{\gamma \gamma_2}|^2 |g_1^{\gamma_2 \gamma'}| \eta_{\gamma_1 \gamma_2} \eta_{\gamma_1 \gamma'} \eta_{\gamma \gamma'}.
\end{aligned} \tag{A12}$$

$$\begin{aligned}
A_4^{\gamma\gamma'}(ncn, c) &= \sum_{\gamma_1, \dots, \gamma_5} \left[\sum_{i=1}^5 (-1)^{i-1} \frac{e^{-iE_{\gamma_i} t}}{d_i(E[\gamma, 4])} \right] \left[\prod_{j=1}^4 g_1^{\gamma_j \gamma_{j+1}} \right] \eta_{\gamma_1 \gamma_3} \delta_{\gamma_2 \gamma_4} \eta_{\gamma_3 \gamma_5} \delta_{\gamma_1 \gamma} \delta_{\gamma_5 \gamma'} \delta_{\gamma \gamma'} \\
&= \sum_{\gamma_1, \gamma_2} \left[-\frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma_2})^2} - \frac{2e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^3 (E_{\gamma} - E_{\gamma_2})} - \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2})^2} \right. \\
&\quad + \frac{2e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})^3 (E_{\gamma_1} - E_{\gamma_2})} + \frac{e^{-iE_{\gamma_2} t}}{(E_{\gamma} - E_{\gamma_2})^2 (E_{\gamma_1} - E_{\gamma_2})^2} + (-it) \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma_2})} \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2})} \right] |g_1^{\gamma \gamma_1}|^2 |g_1^{\gamma_1 \gamma_2}|^2 \eta_{\gamma \gamma_2} \delta_{\gamma \gamma'}. \tag{A13}
\end{aligned}$$

$$\begin{aligned}
A_4^{\gamma\gamma'}(ncn, n) &= \sum_{\gamma_1, \dots, \gamma_5} \left[\sum_{i=1}^5 (-1)^{i-1} \frac{e^{-iE_{\gamma_i} t}}{d_i(E[\gamma, 4])} \right] \left[\prod_{j=1}^4 g_1^{\gamma_j \gamma_{j+1}} \right] \eta_{\gamma_1 \gamma_3} \delta_{\gamma_2 \gamma_4} \eta_{\gamma_3 \gamma_5} \delta_{\gamma_1 \gamma} \delta_{\gamma_5 \gamma'} \eta_{\gamma \gamma'} \\
&= \sum_{\gamma_1, \gamma_2} \left[\frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma_2}) (E_{\gamma} - E_{\gamma'})^2} + \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1}) (E_{\gamma_1} - E_{\gamma_2}) (E_{\gamma_1} - E_{\gamma'})^2} \right. \\
&\quad + \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1}) (E_{\gamma_1} - E_{\gamma_2})^2 (E_{\gamma_1} - E_{\gamma'})} - \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2}) (E_{\gamma_1} - E_{\gamma'})} \\
&\quad - \frac{e^{-iE_{\gamma_2} t}}{(E_{\gamma} - E_{\gamma_2}) (E_{\gamma_1} - E_{\gamma_2})^2 (E_{\gamma_2} - E_{\gamma'})} + \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'})^2 (E_{\gamma_2} - E_{\gamma'})} \\
&\quad \left. - (-it) \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1}) (E_{\gamma_1} - E_{\gamma_2}) (E_{\gamma_1} - E_{\gamma'})} \right] |g_1^{\gamma_1 \gamma_2}|^2 g_1^{\gamma \gamma_1} g_1^{\gamma_1 \gamma'} \eta_{\gamma \gamma_2} \eta_{\gamma_2 \gamma'} \eta_{\gamma \gamma'}. \tag{A14}
\end{aligned}$$

$$\begin{aligned}
A_4^{\gamma\gamma'}(nnc, ck) &= \sum_{\gamma_1, \dots, \gamma_5} \left[\sum_{i=1}^5 (-1)^{i-1} \frac{e^{-iE_{\gamma_i} t}}{d_i(E[\gamma, 4])} \right] \left[\prod_{j=1}^4 g_1^{\gamma_j \gamma_{j+1}} \right] \eta_{\gamma_1 \gamma_3} \eta_{\gamma_2 \gamma_4} \delta_{\gamma_3 \gamma_5} \delta_{\gamma_1 \gamma_4} \delta_{\gamma_1 \gamma} \delta_{\gamma_5 \gamma'} \\
&= \sum_{\gamma_1} \left[-\frac{2e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1}) (E_{\gamma} - E_{\gamma'})^3} - \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma'})^2} + \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma'})^2} \right. \\
&\quad - \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})^2} - \frac{2e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'})^3 (E_{\gamma_1} - E_{\gamma'})} + (-it) \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1}) (E_{\gamma} - E_{\gamma'})^2} \\
&\quad \left. - (-it) \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})} \right] |g_1^{\gamma \gamma'}|^2 g_1^{\gamma \gamma_1} g_1^{\gamma_1 \gamma'}. \tag{A15}
\end{aligned}$$

$$\begin{aligned}
A_4^{\gamma\gamma'}(nnc, nk) &= \sum_{\gamma_1, \dots, \gamma_5} \left[\sum_{i=1}^5 (-1)^{i-1} \frac{e^{-iE_{\gamma_i} t}}{d_i(E[\gamma, 4])} \right] \left[\prod_{j=1}^4 g_1^{\gamma_j \gamma_{j+1}} \right] \eta_{\gamma_1 \gamma_3} \eta_{\gamma_2 \gamma_4} \delta_{\gamma_3 \gamma_5} \eta_{\gamma_1 \gamma_4} \delta_{\gamma_1 \gamma} \delta_{\gamma_5 \gamma'} \\
&= \sum_{\gamma_1, \gamma_2} \left[\frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1}) (E_{\gamma} - E_{\gamma_2}) (E_{\gamma} - E_{\gamma'})^2} - \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1}) (E_{\gamma_1} - E_{\gamma_2}) (E_{\gamma_1} - E_{\gamma'})^2} \right. \\
&\quad + \frac{e^{-iE_{\gamma_2} t}}{(E_{\gamma} - E_{\gamma_2}) (E_{\gamma_1} - E_{\gamma_2}) (E_{\gamma_2} - E_{\gamma'})^2} - \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})^2} \\
&\quad - \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'})^2 (E_{\gamma_2} - E_{\gamma'})} - \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})} \\
&\quad \left. - (-it) \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})} \right] |g_1^{\gamma_2 \gamma'}|^2 g_1^{\gamma \gamma_1} g_1^{\gamma_1 \gamma'} \eta_{\gamma \gamma_2} \eta_{\gamma_1 \gamma_2} \eta_{\gamma \gamma'}. \tag{A16}
\end{aligned}$$

Finally, we calculate the $A_4^{\gamma\gamma'}(nnn)$ by considering the two second decompositions, that is, its former three terms

$$\begin{aligned}
A_4^{\gamma\gamma'}(nnn, cc) &= \sum_{\gamma_1, \dots, \gamma_5} \left[\sum_{i=1}^5 (-1)^{i-1} \frac{e^{-iE_{\gamma_i} t}}{d_i(E[\gamma, 4])} \right] \left[\prod_{j=1}^4 g_1^{\gamma_j \gamma_{j+1}} \right] \left(\prod_{k=1}^3 \eta_{\gamma_k \gamma_{k+2}} \right) \delta_{\gamma_1 \gamma_4} \delta_{\gamma_2 \gamma_5} \delta_{\gamma_1 \gamma} \delta_{\gamma_5 \gamma'} \\
&= \sum_{\gamma_1} \left[-\frac{2e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma'})^3} - \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma'})^2} + \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma'})^2} \right. \\
&\quad - \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})^2} - \frac{2e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'})^3 (E_{\gamma_1} - E_{\gamma'})} + (-it) \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma'})^2} \\
&\quad \left. - (-it) \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})} \right] g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1 \gamma} g_1^{\gamma\gamma'}. \tag{A17}
\end{aligned}$$

$$\begin{aligned}
A_4^{\gamma\gamma'}(nnn, cn) &= \sum_{\gamma_1, \dots, \gamma_5} \left[\sum_{i=1}^5 (-1)^{i-1} \frac{e^{-iE_{\gamma_i} t}}{d_i(E[\gamma, 4])} \right] \left[\prod_{j=1}^4 g_1^{\gamma_j \gamma_{j+1}} \right] \left(\prod_{k=1}^3 \eta_{\gamma_k \gamma_{k+2}} \right) \delta_{\gamma_1 \gamma_4} \eta_{\gamma_2 \gamma_5} \delta_{\gamma_1 \gamma} \delta_{\gamma_5 \gamma'} \\
&= \sum_{\gamma_1, \gamma_2} \left[-\frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma'})^2} - \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})^2 (E_{\gamma} - E_{\gamma'})} \right. \\
&\quad - \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma'})} + \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma'})} \\
&\quad - \frac{e^{-iE_{\gamma_2} t}}{(E_{\gamma} - E_{\gamma_2})^2 (E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_2} - E_{\gamma'})} + \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})} \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma'})} \right] g_1^{\gamma\gamma_1} g_1^{\gamma_1 \gamma_2} g_1^{\gamma_2 \gamma} g_1^{\gamma\gamma'} \eta_{\gamma_1 \gamma'} \eta_{\gamma_2 \gamma'}. \tag{A18}
\end{aligned}$$

$$\begin{aligned}
A_4^{\gamma\gamma'}(nnn, nc) &= \sum_{\gamma_1, \dots, \gamma_5} \left[\sum_{i=1}^5 (-1)^{i-1} \frac{e^{-iE_{\gamma_i} t}}{d_i(E[\gamma, 4])} \right] \left[\prod_{j=1}^4 g_1^{\gamma_j \gamma_{j+1}} \right] \left(\prod_{k=1}^3 \eta_{\gamma_k \gamma_{k+2}} \right) \eta_{\gamma_1 \gamma_4} \delta_{\gamma_2 \gamma_5} \delta_{\gamma_1 \gamma} \delta_{\gamma_5 \gamma'} \\
&= \sum_{\gamma_1, \gamma_2} \left[\frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma'})^2} - \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma'})^2} \right. \\
&\quad + \frac{e^{-iE_{\gamma_2} t}}{(E_{\gamma} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_2} - E_{\gamma'})^2} - \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})^2} \\
&\quad - \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'})^2 (E_{\gamma_2} - E_{\gamma'})} - \frac{e^{-iE_{\gamma'} t}}{(E_{\gamma} - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})} \\
&\quad \left. - (-it) \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})} \right] g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1 \gamma_2} g_1^{\gamma_2 \gamma'} \eta_{\gamma_1 \gamma} \eta_{\gamma_2 \gamma}. \tag{A19}
\end{aligned}$$

while the fourth term has the third decomposition, that is

$$\begin{aligned}
A_4^{\gamma\gamma'}(nnn, nn, c) &= \sum_{\gamma_1, \dots, \gamma_5} \left[\sum_{i=1}^5 (-1)^{i-1} \frac{e^{-iE_{\gamma_i} t}}{d_i(E[\gamma, 4])} \right] \left[\prod_{j=1}^4 g_1^{\gamma_j \gamma_{j+1}} \right] \left(\prod_{k=1}^3 \eta_{\gamma_k \gamma_{k+2}} \right) \delta_{\gamma_1 \gamma} \delta_{\gamma_5 \gamma'} \delta_{\gamma \gamma'} \\
&= \sum_{\gamma_1, \gamma_2, \gamma_3} \left[-\frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma_3})^2} - \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})^2 (E_{\gamma} - E_{\gamma_3})} \right. \\
&\quad - \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma_3})} + \frac{e^{-iE_{\gamma_1} t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma_3})} \\
&\quad - \frac{e^{-iE_{\gamma_2} t}}{(E_{\gamma} - E_{\gamma_2})^2 (E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_2} - E_{\gamma_3})} + \frac{e^{-iE_{\gamma_3} t}}{(E_{\gamma} - E_{\gamma_3})^2 (E_{\gamma_1} - E_{\gamma_3})(E_{\gamma_2} - E_{\gamma_3})} \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma} t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma_3})} \right] g_1^{\gamma\gamma_1} g_1^{\gamma_1 \gamma_2} g_1^{\gamma_2 \gamma_3} g_1^{\gamma_3 \gamma} \eta_{\gamma_2 \gamma} \eta_{\gamma_1 \gamma_3} \delta_{\gamma \gamma'}. \tag{A20}
\end{aligned}$$

$$\begin{aligned}
A_4^{\gamma\gamma'}(nnn, nn, n) &= \sum_{\gamma_1, \dots, \gamma_5} \left[\sum_{i=1}^5 (-1)^{i-1} \frac{e^{-iE_{\gamma_i}t}}{d_i(E[\gamma, 4])} \right] \left[\prod_{j=1}^4 g_1^{\gamma_j \gamma_{j+1}} \right] \left(\prod_{k=1}^3 \eta_{\gamma_k \gamma_{k+2}} \right) \delta_{\gamma_1 \gamma} \delta_{\gamma_5 \gamma'} \eta_{\gamma \gamma'} \\
&= \sum_{\gamma_1, \gamma_2, \gamma_3} \left[\frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma_3})(E_{\gamma} - E_{\gamma'})} \right. \\
&\quad - \frac{e^{-iE_{\gamma_1}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma_3})(E_{\gamma_1} - E_{\gamma'})} \\
&\quad + \frac{e^{-iE_{\gamma_2}t}}{(E_{\gamma} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_2} - E_{\gamma_3})(E_{\gamma_2} - E_{\gamma'})} \\
&\quad - \frac{e^{-iE_{\gamma_3}t}}{(E_{\gamma} - E_{\gamma_3})(E_{\gamma_1} - E_{\gamma_3})(E_{\gamma_2} - E_{\gamma_3})(E_{\gamma_3} - E_{\gamma'})} \\
&\quad \left. + \frac{e^{-iE_{\gamma'}t}}{(E_{\gamma} - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'}) (E_{\gamma_3} - E_{\gamma'})} \right] \\
&\quad \times g_1^{\gamma \gamma_1} g_1^{\gamma_1 \gamma_2} g_1^{\gamma_2 \gamma_3} g_1^{\gamma_3 \gamma'} \eta_{\gamma \gamma_2} \eta_{\gamma \gamma_3} \eta_{\gamma \gamma'} \eta_{\gamma_1 \gamma_3} \eta_{\gamma_1 \gamma'} \eta_{\gamma_2 \gamma'}. \tag{A21}
\end{aligned}$$

Now, all 15 contractions and/or anti-contractions in the fourth order approximation have been calculated out.

In order to absorb the contributions from the fourth order approximation to the improved forms of lower order perturbed solutions, we first decompose $A_4^{\gamma\gamma'}$, which is a summation of all above terms, into the three parts according to their factor forms in $e^{-iE_{\gamma_i}t}$, $(-it)e^{-iE_{\gamma_i}t}$ and $(-it)^2e^{-iE_{\gamma_i}t}/2$, that is

$$A_4^{\gamma\gamma'} = A_4^{\gamma\gamma'}(e) + A_4^{\gamma\gamma'}(te) + A_4^{\gamma\gamma'}(t^2e). \tag{A22}$$

Secondly, we decompose its every term into three parts according to the factor forms in $e^{-iE_{\gamma}t}$, $e^{-iE_{\gamma_1}t}$ ($\sum_{\gamma_1} e^{-iE_{\gamma_1}t}$) and $e^{-iE_{\gamma'}t}$, that is

$$A_4^{\gamma\gamma'}(e) = A_4^{\gamma\gamma'}(e^{-iE_{\gamma}t}) + A_4^{\gamma\gamma'}(e^{-iE_{\gamma_1}t}) + A_4^{\gamma\gamma'}(e^{-iE_{\gamma'}t}), \tag{A23}$$

$$A_4^{\gamma\gamma'}(te) = A_4^{\gamma\gamma'}(te^{-iE_{\gamma}t}) + A_4^{\gamma\gamma'}(te^{-iE_{\gamma_1}t}) + A_4^{\gamma\gamma'}(te^{-iE_{\gamma'}t}), \tag{A24}$$

$$A_4^{\gamma\gamma'}(t^2e) = A_4^{\gamma\gamma'}(t^2e^{-iE_{\gamma}t}) + A_4^{\gamma\gamma'}(t^2e^{-iE_{\gamma_1}t}) + A_4^{\gamma\gamma'}(t^2e^{-iE_{\gamma'}t}). \tag{A25}$$

Finally, we again decompose every term in the above equations into the diagonal and off-diagonal parts about γ and γ' , that is

$$A_4^{\gamma\gamma'}(e^{-iE_{\gamma_i}t}) = A_4^{\gamma\gamma'}(e^{-iE_{\gamma_i}t}; D) + A_4^{\gamma\gamma'}(e^{-iE_{\gamma_i}t}; N), \tag{A26}$$

$$A_4^{\gamma\gamma'}(te^{-iE_{\gamma_i}t}) = A_4^{\gamma\gamma'}(te^{-iE_{\gamma_i}t}; D) + A_4^{\gamma\gamma'}(te^{-iE_{\gamma_i}t}; N), \tag{A27}$$

$$A_4^{\gamma\gamma'}(t^2e^{-iE_{\gamma_i}t}) = A_4^{\gamma\gamma'}(t^2e^{-iE_{\gamma_i}t}; D) + A_4^{\gamma\gamma'}(t^2e^{-iE_{\gamma_i}t}; N). \tag{A28}$$

where E_{γ_i} takes E_{γ} , E_{γ_1} and $E_{\gamma'}$.

If we do not concern the improved forms of perturbed solutions equal to or higher than the fourth order one, we only need to write down the second and third terms in eq.(A22) and calculate their diagonal and off-diagonal parts respectively. Based on the calculated results above, it is easy to obtain

$$\begin{aligned}
A_4^{\gamma\gamma'}(te^{-iE_{\gamma}t}; D) &= (-it)e^{-iE_{\gamma}t} \left[\sum_{\gamma_1} \frac{-2|g_1^{\gamma\gamma_1}|^4}{(E_{\gamma} - E_{\gamma_1})^3} - \sum_{\gamma_1, \gamma_2} \frac{|g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma_2}|^2 \eta_{\gamma_1 \gamma_2}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})^2} \right. \\
&\quad - \sum_{\gamma_1, \gamma_2} \frac{|g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma_2}|^2 \eta_{\gamma_1 \gamma_2}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma_2})} + \sum_{\gamma_1, \gamma_2} \frac{|g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma_1 \gamma_2}|^2 \eta_{\gamma \gamma_2}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma_2})} \\
&\quad \left. + \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{g_1^{\gamma \gamma_1} g_1^{\gamma_1 \gamma_2} g_1^{\gamma_2 \gamma_3} g_1^{\gamma_3 \gamma'} \eta_{\gamma \gamma_2} \eta_{\gamma_1 \gamma_3}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma_3})} \right] \delta_{\gamma \gamma'}. \tag{A29}
\end{aligned}$$

Substituting the relation $\eta_{\beta_1 \beta_2} = 1 - \delta_{\beta_1 \beta_2}$, using the technology of index exchanging and introducing the definitions of so-called the a th revision energy $G_{\gamma}^{(a)}$:

$$G_{\gamma}^{(2)} = \sum_{\gamma_1} \frac{|g_1^{\gamma \gamma_1}|^2}{E_{\gamma} - E_{\gamma_1}} \tag{A30}$$

$$G_\gamma^{(4)} = \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma} \eta_{\gamma\gamma_2}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})} - \sum_{\gamma_1, \gamma_2} \frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2})}, \quad (\text{A31})$$

we can simplify Eq. (A29) to the following concise form:

$$A_4^{\gamma\gamma'}(te^{-iE_\gamma t}; D) = -(-it)e^{-iE_\gamma t} \left[\sum_{\gamma_1} \frac{G_\gamma^{(2)}}{(E_\gamma - E_{\gamma_1})^2} |g_1^{\gamma\gamma_1}|^2 - G_\gamma^{(4)} \right] \delta_{\gamma\gamma'}. \quad (\text{A32})$$

Similar calculation and simplification lead to

$$A_4^{\gamma\gamma'}(te^{-iE_\gamma t}; N) = (-it)e^{-iE_\gamma t} \left[\frac{G_\gamma^{(3)} g_1^{\gamma\gamma'}}{(E_\gamma - E_{\gamma'})} + \sum_{\gamma_1} \frac{G_\gamma^{(2)}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma'})} \right], \quad (\text{A33})$$

where

$$G_\gamma^{(3)} = \sum_{\gamma_1, \gamma_2} \frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})}. \quad (\text{A34})$$

For saving the space, the corresponding detail is omitted. In fact, it is not difficult, but it is necessary to be careful enough, specially in the cases of higher order approximations.

In the same way, we can obtain:

$$A_4^{\gamma\gamma'}(te^{-iE_{\gamma_1} t}; D) = (-it) \sum_{\gamma_1} \frac{G_{\gamma_1}^{(2)} e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^2} |g_1^{\gamma\gamma_1}|^2 \delta_{\gamma\gamma'}, \quad (\text{A35})$$

$$A_4^{\gamma\gamma'}(te^{-iE_{\gamma_1} t}; N) = -(-it) \sum_{\gamma_1} \frac{G_{\gamma_1}^{(2)} e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma'})} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'}. \quad (\text{A36})$$

$$A_4^{\gamma\gamma'}(te^{-iE_{\gamma'} t}; D) = 0, \quad (\text{A37})$$

$$A_4^{\gamma\gamma'}(te^{-iE_{\gamma'} t}; N) = -(-it)e^{-iE_{\gamma'} t} \left[\sum_{\gamma_1} \frac{G_{\gamma'}^{(3)}}{(E_\gamma - E_{\gamma'})} g_1^{\gamma\gamma'} \right. \\ \left. + \sum_{\gamma_1} \frac{G_{\gamma'}^{(2)}}{(E_\gamma - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'} \right]. \quad (\text{A38})$$

For the terms with the factor $t^2 e$, only one is nonzero, that is

$$A_4^{\gamma\gamma'}(t^2 e) = A_4^{\gamma\gamma'}(t^2 e^{-iE_\gamma t}; D) = \frac{(-iG_\gamma^{(2)} t)^2}{2!} e^{-iE_\gamma t}. \quad (\text{A39})$$

since

$$A_4^{\gamma\gamma'}(t^2 e^{-iE_{\gamma_1} t}; D) = A_4^{\gamma\gamma'}(t^2 e^{-iE_{\gamma'} t}; D) = 0, \quad (\text{A40})$$

$$A_4^{\gamma\gamma'}(t^2 e^{-iE_\gamma t}; N) = A_4^{\gamma\gamma'}(t^2 e^{-iE_{\gamma_1} t}; D) = A_4^{\gamma\gamma'}(t^2 e^{-iE_{\gamma'} t}; D) = 0. \quad (\text{A41})$$

We can see that these terms can be absorbed into (or merged with) the lower order approximations to obtain the improved forms of perturbed solutions.

2. $l=5$ case

Now let we consider the case of the fifth order approximation ($l = 5$). From eq.(42) it follows that the first decompositions of g -product have $2^4 = 16$ terms. They can be divided into 5 groups

$$A_5^{\gamma\gamma'} = \sum_{i=0}^4 \mathcal{A}_5^{\gamma\gamma'}(i; \eta), \quad (\text{A42})$$

where i indicates the number of η functions. Obviously

$$\mathcal{A}_5^{\gamma\gamma'}(0; \eta) = A_5^{\gamma\gamma'}(cccc), \quad (\text{A43})$$

$$\mathcal{A}_5^{\gamma\gamma'}(1; \eta) = A_5^{\gamma\gamma'}(cccn) + A_5^{\gamma\gamma'}(ccnc) + A_5^{\gamma\gamma'}(cncc) + A_5^{\gamma\gamma'}(nccc), \quad (\text{A44})$$

$$\begin{aligned} \mathcal{A}_5^{\gamma\gamma'}(2; \eta) = & A_5^{\gamma\gamma'}(ccnn) + A_5^{\gamma\gamma'}(cncn) + A_5^{\gamma\gamma'}(cnnc) \\ & + A_5^{\gamma\gamma'}(nccn) + A_5^{\gamma\gamma'}(ncnc) + A_5^{\gamma\gamma'}(nncc), \end{aligned} \quad (\text{A45})$$

$$\mathcal{A}_5^{\gamma\gamma'}(3; \eta) = A_5^{\gamma\gamma'}(cnnn) + A_5^{\gamma\gamma'}(ncnn) + A_5^{\gamma\gamma'}(nncn) + A_5^{\gamma\gamma'}(nnnc), \quad (\text{A46})$$

$$\mathcal{A}_5^{\gamma\gamma'}(4; \eta) = A_5^{\gamma\gamma'}(nnnn). \quad (\text{A47})$$

Here, we have used the notations stated in Sec. VI.

By calculation, we obtain the $\mathcal{A}_5^{\gamma\gamma'}(0, \eta)$ and every term of $\mathcal{A}_5^{\gamma\gamma'}(1, \eta)$ have only nontrivial first contractions and/or anti-contractions. But, we can find that every term of $\mathcal{A}_5^{\gamma\gamma'}(2, \eta)$ can have one nontrivial second or third or fourth contraction or anti-contraction, that is

$$A_5^{\gamma\gamma'}(ccnn) = A_5^{\gamma\gamma'}(ccnn, kkc) + A_5^{\gamma\gamma'}(ccnn, kkn), \quad (\text{A48})$$

$$A_5^{\gamma\gamma'}(cncn) = A_5^{\gamma\gamma'}(cncn, kc) + A_5^{\gamma\gamma'}(cncn, kn), \quad (\text{A49})$$

$$A_5^{\gamma\gamma'}(cnnc) = A_5^{\gamma\gamma'}(cnnc, cck) + A_5^{\gamma\gamma'}(cnnc, ckn), \quad (\text{A50})$$

$$A_5^{\gamma\gamma'}(nccn) = A_5^{\gamma\gamma'}(nccn, c) + A_5^{\gamma\gamma'}(nccn, n), \quad (\text{A51})$$

$$A_5^{\gamma\gamma'}(ncnc) = A_5^{\gamma\gamma'}(ncnc, ck) + A_5^{\gamma\gamma'}(ncnc, nk), \quad (\text{A52})$$

$$A_5^{\gamma\gamma'}(nncc) = A_5^{\gamma\gamma'}(nncc, ckk) + A_5^{\gamma\gamma'}(nncc, nkk). \quad (\text{A53})$$

Similarly, every term of $\mathcal{A}_5^{\gamma\gamma'}(3, \eta)$ can have two higher order contractions and/or anti-contractions:

$$\begin{aligned} A_5^{\gamma\gamma'}(cnnn) = & A_5^{\gamma\gamma'}(cnnn, kcc) + A_5^{\gamma\gamma'}(cnnn, kcn) \\ & + A_5^{\gamma\gamma'}(cnnn, kn) + A_5^{\gamma\gamma'}(cnnn, knn), \end{aligned} \quad (\text{A54})$$

$$\begin{aligned} A_5^{\gamma\gamma'}(ncnn) = & A_5^{\gamma\gamma'}(ncnn, kkc, ck) + A_5^{\gamma\gamma'}(ncnn, kkn, ck) \\ & + A_5^{\gamma\gamma'}(ncnn, kkc, nk) + A_5^{\gamma\gamma'}(ncnn, kkn, nk), \end{aligned} \quad (\text{A55})$$

$$\begin{aligned} A_5^{\gamma\gamma'}(nncn) = & A_5^{\gamma\gamma'}(nncn, ckk, kc) + A_5^{\gamma\gamma'}(nncn, ckk, kn) \\ & + A_5^{\gamma\gamma'}(nncn, nkk, kc) + A_5^{\gamma\gamma'}(nncn, nkk, kn), \end{aligned} \quad (\text{A56})$$

$$\begin{aligned} A_5^{\gamma\gamma'}(nnnc) = & A_5^{\gamma\gamma'}(nnnc, cck) + A_5^{\gamma\gamma'}(nnnc, cnk) \\ & + A_5^{\gamma\gamma'}(nnnc, nck) + A_5^{\gamma\gamma'}(nnnc, nnk). \end{aligned} \quad (\text{A57})$$

Moreover, their last terms, with two higher order anti-contractions, can have one nontrivial more higher contraction or anti-contraction:

$$A_5^{\gamma\gamma'}(cnnn, knn) = A_5^{\gamma\gamma'}(cnnn, knn, kc) + A_5^{\gamma\gamma'}(cnnn, knn, kn), \quad (\text{A58})$$

$$A_5^{\gamma\gamma'}(ncnn, kkn, nk) = A_5^{\gamma\gamma'}(ncnn, kkn, nk, c) + A_5^{\gamma\gamma'}(ncnn, kkn, nk, n), \quad (\text{A59})$$

$$A_5^{\gamma\gamma'}(nncn, nkk, kn) = A_5^{\gamma\gamma'}(nncn, nkk, kn, c) + A_5^{\gamma\gamma'}(nncn, nkk, kn, n), \quad (\text{A60})$$

$$A_5^{\gamma\gamma'}(nnnc, nnk) = A_5^{\gamma\gamma'}(nnnc, nnk, ck) + A_5^{\gamma\gamma'}(nnnc, nnk, nk). \quad (\text{A61})$$

In the case of $A_5^{\gamma\gamma'}(nnnn)$, there are three terms corresponding to the second decompositions that result in

$$\begin{aligned} A_5^{\gamma\gamma'}(nnnn) = & A_5^{\gamma\gamma'}(nnnn, ccc) + A_5^{\gamma\gamma'}(nnnn, ccn) + A_5^{\gamma\gamma'}(nnnn, cnc) + A_5^{\gamma\gamma'}(nnnn, ncc) \\ & + A_5^{\gamma\gamma'}(nnnn, cnn) + A_5^{\gamma\gamma'}(nnnn, ncn) + A_5^{\gamma\gamma'}(nnnn, nnc) + A_5^{\gamma\gamma'}(nnnn, nnn). \end{aligned} \quad (\text{A62})$$

In the above expression, from the fifth term to the seventh term have the third- or fourth- contraction and anti-contraction, the eighth term has two third contractions and anti-contractions:

$$A_5^{\gamma\gamma'}(nnnn, cnn) = A_5^{\gamma\gamma'}(nnnn, cnn, kc) + A_5^{\gamma\gamma'}(nnnn, cnn, kn), \quad (\text{A63})$$

$$A_5^{\gamma\gamma'}(nnnn, ncn) = A_5^{\gamma\gamma'}(nnnn, ncn, c) + A_5^{\gamma\gamma'}(nnnn, ncn, n), \quad (\text{A64})$$

$$A_5^{\gamma\gamma'}(nnnn, nnc) = A_5^{\gamma\gamma'}(nnnn, nnc, ck) + A_5^{\gamma\gamma'}(nnnn, nnc, nk), \quad (\text{A65})$$

$$A_5^{\gamma\gamma'}(nnnn, nnn) = A_5^{\gamma\gamma'}(nnnn, nnn, cc) + A_5^{\gamma\gamma'}(nnnn, nnn, cn) \\ + A_5^{\gamma\gamma'}(nnnn, nnn, nc) + A_5^{\gamma\gamma'}(nnnn, nnn, nn). \quad (\text{A66})$$

In addition, $A_5^{\gamma\gamma'}(nnnn, nnn, nn)$ consists of the fourth contraction and anti-contraction

$$A_5^{\gamma\gamma'}(nnnn, nnn, nn) = A_5^{\gamma\gamma'}(nnnn, nnn, nn, c) + A_5^{\gamma\gamma'}(nnnn, nnn, nn, n). \quad (\text{A67})$$

According to the above analysis, we obtain that the contribution from the five order approximation is made of 52 terms after finding out all of contractions and anti-contractions.

Just like we have done in the $l = 4$ case, we decompose

$$A_5^{\gamma\gamma'} = A_5^{\gamma\gamma'}(e) + A_5^{\gamma\gamma'}(te) + A_5^{\gamma\gamma'}(t^2e), \quad (\text{A68})$$

where

$$A_5^{\gamma\gamma'}(e) = A_5^{\gamma\gamma'}(e^{-iE_\gamma t}) + A_5^{\gamma\gamma'}(e^{-iE_{\gamma_1} t}) + A_5^{\gamma\gamma'}(e^{-iE_{\gamma_2} t}) + A_5^{\gamma\gamma'}(e^{-iE_{\gamma'} t}), \quad (\text{A69})$$

$$A_4^{\gamma\gamma'}(te) = A_5^{\gamma\gamma'}(te^{-iE_\gamma t}) + A_5^{\gamma\gamma'}(te^{-iE_{\gamma_1} t}) + A_5^{\gamma\gamma'}(te^{-iE_{\gamma_2} t}) + A_5^{\gamma\gamma'}(te^{-iE_{\gamma'} t}), \quad (\text{A70})$$

$$A_5^{\gamma\gamma'}(t^2e) = A_5^{\gamma\gamma'}(t^2e^{-iE_\gamma t}) + A_5^{\gamma\gamma'}(t^2e^{-iE_{\gamma_1} t}) + A_5^{\gamma\gamma'}(t^2e^{-iE_{\gamma_2} t}) + A_5^{\gamma\gamma'}(t^2e^{-iE_{\gamma'} t}). \quad (\text{A71})$$

While, every term in the above equations has its diagonal and off-diagonal parts about γ and γ' , that is

$$A_5^{\gamma\gamma'}(e^{-iE_{\gamma_i} t}) = A_5^{\gamma\gamma'}(e^{-iE_{\gamma_i} t}; D) + A_5^{\gamma\gamma'}(e^{-iE_{\gamma_i} t}; N), \quad (\text{A72})$$

$$A_5^{\gamma\gamma'}(te^{-iE_{\gamma_i} t}) = A_5^{\gamma\gamma'}(te^{-iE_{\gamma_i} t}; D) + A_5^{\gamma\gamma'}(te^{-iE_{\gamma_i} t}; N), \quad (\text{A73})$$

$$A_5^{\gamma\gamma'}(t^2e^{-iE_{\gamma_i} t}) = A_5^{\gamma\gamma'}(t^2e^{-iE_{\gamma_i} t}; D) + A_5^{\gamma\gamma'}(t^2e^{-iE_{\gamma_i} t}; N). \quad (\text{A74})$$

where E_{γ_i} takes $E_\gamma, E_{\gamma_1}, E_{\gamma_2}$ and $E_{\gamma'}$.

If we do not concern the improved forms of perturbed solution higher than the fourth order one, we only need to write down the second and third terms in eq.(A68). We can calculate them and the results are put in the supplementary of Ref. [10].

Based on these contraction- and anti contraction- expressions, we can, in terms of the rearrangement and summation, obtain

$$A_5(te^{-iE_\gamma t}, D) = -(-iG_\gamma^{(3)}t) \sum_{\gamma_1} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma} \delta_{\gamma\gamma'} \\ - (-iG_\gamma^{(2)}t) \sum_{\gamma_1, \gamma_2} \left[\frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2})} \right. \\ \left. + \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2})^2} \right] g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} \delta_{\gamma\gamma'} + (-iG_\gamma^{(5)}t) \delta_{\gamma\gamma'} \quad (\text{A75})$$

where

$$G_\gamma^{(5)} = \sum_{\gamma_1, \gamma_2, \gamma_3, \gamma_4} \frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma_4} g_1^{\gamma_4\gamma} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma_3}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma_3}) (E_\gamma - E_{\gamma_4})} \\ - \sum_{\gamma_1, \gamma_2, \gamma_3} \left[\frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_1\gamma} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma_3})} + \frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_1\gamma} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2})^2 (E_\gamma - E_{\gamma_3})} \right. \\ \left. + \frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_1\gamma} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma_3})^2} \right]. \quad (\text{A76})$$

$$\begin{aligned}
A_5(te^{-iE_{\gamma_1}t}, D) &= (-iG_{\gamma}^{(3)}t) \sum_{\gamma_1} \frac{e^{-iE_{\gamma_1}t}}{(E_{\gamma} - E_{\gamma_1})^2} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma} \delta_{\gamma\gamma'} \\
&\quad + (-iG_{\gamma}^{(2)}t) \sum_{\gamma_1, \gamma_2} \frac{e^{-iE_{\gamma_1}t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2})} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} \delta_{\gamma\gamma'}
\end{aligned} \tag{A77}$$

$$A_5(te^{-iE_{\gamma_2}t}, D) = -(-iG_{\gamma}^{(2)}t) \sum_{\gamma_1, \gamma_2} \frac{e^{-iE_{\gamma_2}t}}{(E_{\gamma} - E_{\gamma_2})^2 (E_{\gamma_1} - E_{\gamma_2})} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} \delta_{\gamma\gamma'} \tag{A78}$$

$$A_5(te^{-iE_{\gamma'}t}, D) = 0 \tag{A79}$$

$$\begin{aligned}
A_5(te^{-iE_{\gamma}t}, N) &= (-iG_{\gamma}^{(4)}t) \frac{e^{-iE_{\gamma}t} g_1^{\gamma\gamma'}}{(E_{\gamma} - E_{\gamma'})} + (-iG_{\gamma}^{(3)}t) \sum_{\gamma_1} \frac{e^{-iE_{\gamma}t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta^{\gamma\gamma'}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma'})} \\
&\quad - (-iG_{\gamma}^{(2)}t) \sum_{\gamma_1} \left[\frac{e^{-iE_{\gamma}t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma'})} + \frac{e^{-iE_{\gamma}t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma'})^2} \right] \\
&\quad + (-iG_{\gamma}^{(2)}t) \sum_{\gamma_1, \gamma_2} \frac{e^{-iE_{\gamma}t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma'}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma'})}
\end{aligned} \tag{A80}$$

$$\begin{aligned}
A_5(te^{-iE_{\gamma_1}t}, N) &= -(-it) \sum_{\gamma_1} \frac{G_{\gamma_1}^{(3)} e^{-iE_{\gamma_1}t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma'})} \\
&\quad - (-it) \sum_{\gamma_1, \gamma_2} \frac{G_{\gamma_1}^{(2)} e^{-iE_{\gamma_1}t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma_1\gamma'} \eta_{\gamma\gamma'}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma'})}
\end{aligned} \tag{A81}$$

$$A_5(te^{-iE_{\gamma_2}t}, N) = (-it) \sum_{\gamma_1, \gamma_2} \frac{G_{\gamma_2}^{(2)} e^{-iE_{\gamma_2}t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma'}}{(E_{\gamma} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_2} - E_{\gamma'})} \tag{A82}$$

$$\begin{aligned}
A_5(te^{-iE_{\gamma'}t}, N) &= (-iG_{\gamma'}^{(4)}t) \frac{e^{-iE_{\gamma'}t} g_1^{\gamma\gamma'}}{(E_{\gamma} - E_{\gamma'})} + (-iG_{\gamma'}^{(3)}t) \sum_{\gamma_1} \frac{e^{-iE_{\gamma'}t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta^{\gamma\gamma'}}{(E_{\gamma} - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'})} \\
&\quad + (-iG_{\gamma'}^{(2)}t) \sum_{\gamma_1} \left[\frac{e^{-iE_{\gamma'}t} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma'} g_1^{\gamma\gamma'}}{(E_{\gamma} - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})} + \frac{e^{-iE_{\gamma'}t} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma'} g_1^{\gamma\gamma'}}{(E_{\gamma} - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'})^2} \right] \\
&\quad - (-iG_{\gamma'}^{(2)}t) \sum_{\gamma_1, \gamma_2} \frac{e^{-iE_{\gamma'}t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma_1\gamma'} \eta_{\gamma\gamma'}}{(E_{\gamma} - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})}
\end{aligned} \tag{A83}$$

For the parts with t^2e , we have

$$A_5(t^2e^{-iE_{\gamma}t}, D) = \frac{(-it)^2}{2!} 2G_{\gamma}^{(2)} G_{\gamma}^{(3)} \delta_{\gamma\gamma'} e^{-iE_{\gamma}t}, \tag{A84}$$

$$A_5(t^2e^{-iE_{\gamma_1}t}, D) = A_5(t^2e^{-iE_{\gamma_2}t}, D) = A_5(t^2e^{-iE_{\gamma'}t}, D) = 0. \tag{A85}$$

$$A_5(t^2e^{-iE_{\gamma}t}, N) = \frac{(-it)^2}{2!} \left(G_{\gamma}^{(2)}\right)^2 \frac{e^{-iE_{\gamma}t}}{E_{\gamma} - E_{\gamma'}} g_1^{\gamma\gamma'}, \tag{A86}$$

$$A_5(t^2e^{-iE_{\gamma_2}t}, N) = A_5(t^2e^{-iE_{\gamma'}t}, N) = 0, \tag{A87}$$

$$A_5(t^2e^{-iE_{\gamma'}t}, N) = -\frac{(-it)^2}{2!} \left(G_{\gamma'}^{(2)}\right)^2 \frac{e^{-iE_{\gamma'}t}}{E_{\gamma} - E_{\gamma'}} g_1^{\gamma\gamma'}. \tag{A88}$$

It is clear that the above diagonal and off-diagonal part about $A_5^{\gamma\gamma'}(te)$ and $A_5^{\gamma'\gamma}(te)$ indeed has the expected forms and can be absorbed reasonably into the lower order approximations in order to obtain the improved forms of perturbed solutions.

3. $l = 6$ case

Now let we consider the case of the sixth order approximation ($l = 6$). From eq.(42) it follows that the first decompositions of g -product have $2^5 = 32$ terms. Like the $l = 5$ case, they can be divided into 6 groups

$$A_6^{\gamma\gamma'} = \sum_{i=0}^4 \mathcal{A}_6^{\gamma\gamma'}(i; \eta), \quad (\text{A89})$$

where i indicates the number of η functions. Obviously

$$\mathcal{A}_6^{\gamma\gamma'}(0; \eta) = A_6^{\gamma\gamma'}(ccccc), \quad (\text{A90})$$

$$\begin{aligned} \mathcal{A}_6^{\gamma\gamma'}(1; \eta) = & A_6^{\gamma\gamma'}(ccccn) + A_6^{\gamma\gamma'}(cccnec) + A_6^{\gamma\gamma'}(ccncec) \\ & + A_6^{\gamma\gamma'}(cnccc) + A_6^{\gamma\gamma'}(ncccc), \end{aligned} \quad (\text{A91})$$

$$\begin{aligned} \mathcal{A}_6^{\gamma\gamma'}(2; \eta) = & A_6^{\gamma\gamma'}(cccn) + A_6^{\gamma\gamma'}(ccnec) + A_6^{\gamma\gamma'}(cnccn) + A_6^{\gamma\gamma'}(ncccn) \\ & + A_6^{\gamma\gamma'}(ccnnc) + A_6^{\gamma\gamma'}(cnenc) + A_6^{\gamma\gamma'}(nccnc) + A_6^{\gamma\gamma'}(cnccc) \\ & + A_6^{\gamma\gamma'}(ncncc) + A_6^{\gamma\gamma'}(nnccc), \end{aligned} \quad (\text{A92})$$

$$\begin{aligned} \mathcal{A}_6^{\gamma\gamma'}(3; \eta) = & A_6^{\gamma\gamma'}(ccnnn) + A_6^{\gamma\gamma'}(cnenn) + A_6^{\gamma\gamma'}(cnncn) + A_6^{\gamma\gamma'}(cnnnec) \\ & + A_6^{\gamma\gamma'}(nccnn) + A_6^{\gamma\gamma'}(ncnec) + A_6^{\gamma\gamma'}(ncnnc) + A_6^{\gamma\gamma'}(nnccn) \\ & + A_6^{\gamma\gamma'}(nnenc) + A_6^{\gamma\gamma'}(nnncc), \end{aligned} \quad (\text{A93})$$

$$\begin{aligned} \mathcal{A}_6^{\gamma\gamma'}(4; \eta) = & A_6^{\gamma\gamma'}(cnnnn) + A_6^{\gamma\gamma'}(ncnnn) + A_6^{\gamma\gamma'}(nncnn) \\ & + A_6^{\gamma\gamma'}(nnnen) + A_6^{\gamma\gamma'}(nnnnc) \end{aligned} \quad (\text{A94})$$

$$\mathcal{A}_6^{\gamma\gamma'}(5; \eta) = A_6^{\gamma\gamma'}(nnnnn). \quad (\text{A95})$$

Furthermore considering the high order contraction or anti-contraction, we have

$$A_6^{\gamma\gamma'}(ccenn) = A_6^{\gamma\gamma'}(ccenn, kkkc) + A_6^{\gamma\gamma'}(ccenn, kkn), \quad (\text{A96})$$

$$A_6^{\gamma\gamma'}(ccnec) = A_6^{\gamma\gamma'}(ccnec, kkc) + A_6^{\gamma\gamma'}(ccnec, kn), \quad (\text{A97})$$

$$A_6^{\gamma\gamma'}(ccnnc) = A_6^{\gamma\gamma'}(ccnnc, kckc) + A_6^{\gamma\gamma'}(ccnnc, kknk), \quad (\text{A98})$$

$$A_6^{\gamma\gamma'}(cnccn) = A_6^{\gamma\gamma'}(cnccn, kc) + A_6^{\gamma\gamma'}(cnccn, kn), \quad (\text{A99})$$

$$A_6^{\gamma\gamma'}(cnenc) = A_6^{\gamma\gamma'}(cnenc, kck) + A_6^{\gamma\gamma'}(cnenc, knk), \quad (\text{A100})$$

$$A_6^{\gamma\gamma'}(cnccc) = A_6^{\gamma\gamma'}(cnccc, kckk) + A_6^{\gamma\gamma'}(cnccc, knkk), \quad (\text{A101})$$

$$A_6^{\gamma\gamma'}(ncccn) = A_6^{\gamma\gamma'}(ncccn, c) + A_6^{\gamma\gamma'}(ncccn, n), \quad (\text{A102})$$

$$A_6^{\gamma\gamma'}(ncnec) = A_6^{\gamma\gamma'}(ncnec, ck) + A_6^{\gamma\gamma'}(ncnec, nk), \quad (\text{A103})$$

$$A_6^{\gamma\gamma'}(ncncc) = A_6^{\gamma\gamma'}(ncncc, ckk) + A_6^{\gamma\gamma'}(ncncc, nkk), \quad (\text{A104})$$

$$A_6^{\gamma\gamma'}(nnccc) = A_6^{\gamma\gamma'}(nnccc, ckck) + A_6^{\gamma\gamma'}(nnccc, nkkk) \quad (\text{A105})$$

$$\begin{aligned} A_6^{\gamma\gamma'}(ccnnn) = & A_6^{\gamma\gamma'}(ccnnn, kkc) + A_6^{\gamma\gamma'}(ccnnn, kkn) + A_6^{\gamma\gamma'}(ccnnn, knkc) \\ & + A_6^{\gamma\gamma'}(ccnnn, kkn, kkc) + A_6^{\gamma\gamma'}(ccnnn, kkn, kn), \end{aligned} \quad (\text{A106})$$

$$A_6^{\gamma\gamma'}(cncnn) = A_6^{\gamma\gamma'}(cncnn, kkkc, cck) + A_6^{\gamma\gamma'}(cncnn, kkkc, knk) + A_6^{\gamma\gamma'}(cncnn, kkkn, cck) + A_6^{\gamma\gamma'}(cncnn, kkkn, knk, kc) + A_6^{\gamma\gamma'}(cncnn, kkkn, knk, kn), \quad (A107)$$

$$A_6^{\gamma\gamma'}(cnnnc) = A_6^{\gamma\gamma'}(cnnnc, kckk, kkc) + A_6^{\gamma\gamma'}(cnnnc, kckk, knk) + A_6^{\gamma\gamma'}(cnnnc, knkk, kkc) + A_6^{\gamma\gamma'}(cnnnc, knkk, knk, kc) + A_6^{\gamma\gamma'}(cnnnc, knkk, knk, kn), \quad (A108)$$

$$A_6^{\gamma\gamma'}(cnnnc) = A_6^{\gamma\gamma'}(cnnnc, kckk, kkc) + A_6^{\gamma\gamma'}(cnnnc, kcnk) + A_6^{\gamma\gamma'}(cnnnc, knck) + A_6^{\gamma\gamma'}(cnnnc, knnk, cck) + A_6^{\gamma\gamma'}(cnnnc, knnk, knk), \quad (A109)$$

$$A_6^{\gamma\gamma'}(nccnn) = A_6^{\gamma\gamma'}(nccnn, kkkc, ck) + A_6^{\gamma\gamma'}(nccnn, kkkc, nk) + A_6^{\gamma\gamma'}(nccnn, kkkn, ck) + A_6^{\gamma\gamma'}(nccnn, kkkn, nk, c) + A_6^{\gamma\gamma'}(nccnn, kkkn, nk, n), \quad (A110)$$

$$A_6^{\gamma\gamma'}(ncncn) = A_6^{\gamma\gamma'}(ncncn, ckc) + A_6^{\gamma\gamma'}(ncncn, ckn) + A_6^{\gamma\gamma'}(ncncn, nkc) + A_6^{\gamma\gamma'}(ncncn, knk, c) + A_6^{\gamma\gamma'}(ncncn, knk, n), \quad (A111)$$

$$A_6^{\gamma\gamma'}(ncnnc) = A_6^{\gamma\gamma'}(ncnnc, kckc, cck) + A_6^{\gamma\gamma'}(ncnnc, kckc, nkk) + A_6^{\gamma\gamma'}(ncnnc, kknk, cck) + A_6^{\gamma\gamma'}(ncnnc, kknk, nkk, ck) + A_6^{\gamma\gamma'}(ncnnc, kknk, nkk, nk), \quad (A112)$$

$$A_6^{\gamma\gamma'}(nnccn) = A_6^{\gamma\gamma'}(nnccn, cc) + A_6^{\gamma\gamma'}(nnccn, cn) + A_6^{\gamma\gamma'}(nnccn, nc) + A_6^{\gamma\gamma'}(nnccn, nn, c) + A_6^{\gamma\gamma'}(nnccn, nn, n), \quad (A113)$$

$$A_6^{\gamma\gamma'}(nnncn) = A_6^{\gamma\gamma'}(nnncn, ckkk, cck) + A_6^{\gamma\gamma'}(nnncn, ckkk, knk) + A_6^{\gamma\gamma'}(nnncn, nkkk, cck) + A_6^{\gamma\gamma'}(nnncn, nkkk, knk, ck) + A_6^{\gamma\gamma'}(nnncn, nkkk, knk, nk), \quad (A114)$$

$$A_6^{\gamma\gamma'}(nnncc) = A_6^{\gamma\gamma'}(nnncc, cckk) + A_6^{\gamma\gamma'}(nnncc, cnkk) + A_6^{\gamma\gamma'}(nnncc, nckk) + A_6^{\gamma\gamma'}(nnncc, nnkk, c) + A_6^{\gamma\gamma'}(nnncc, nnkk, n), \quad (A115)$$

$$A_6^{\gamma\gamma'}(cnnnn) = A_6^{\gamma\gamma'}(cnnnn, kccc) + A_6^{\gamma\gamma'}(cnnnn, kccn) + A_6^{\gamma\gamma'}(cnnnn, kcnc) + A_6^{\gamma\gamma'}(cnnnn, kncc) + A_6^{\gamma\gamma'}(cnnnn, kcnn) + A_6^{\gamma\gamma'}(cnnnn, kncn) + A_6^{\gamma\gamma'}(cnnnn, knnc) + A_6^{\gamma\gamma'}(cnnnn, knnn), \quad (A116)$$

$$A_6^{\gamma\gamma'}(cnnnn, kcnn) = A_6^{\gamma\gamma'}(cnnnn, kcnn, kkc) + A_6^{\gamma\gamma'}(cnnnn, kcnn, knk), \quad (A117)$$

$$A_6^{\gamma\gamma'}(cnnnn, kncn) = A_6^{\gamma\gamma'}(cnnnn, kncn, kc) + A_6^{\gamma\gamma'}(cnnnn, kncn, kn), \quad (A118)$$

$$A_6^{\gamma\gamma'}(cnnnn, knnc) = A_6^{\gamma\gamma'}(cnnnn, knnc, kck) + A_6^{\gamma\gamma'}(cnnnn, knnc, knk), \quad (A119)$$

$$A_6^{\gamma\gamma'}(cnnnn, knnn) = A_6^{\gamma\gamma'}(cnnnn, knnn, kcc) + A_6^{\gamma\gamma'}(cnnnn, knnn, kcn) + A_6^{\gamma\gamma'}(cnnnn, knnn, knk) + A_6^{\gamma\gamma'}(cnnnn, knnn, knn, kc) + A_6^{\gamma\gamma'}(cnnnn, knnn, knn, kn). \quad (A120)$$

$$A_6^{\gamma\gamma'}(ncnnn) = A_6^{\gamma\gamma'}(ncnnn, kkc, cck) + A_6^{\gamma\gamma'}(ncnnn, kkc, nkk) + A_6^{\gamma\gamma'}(ncnnn, kkc, cck) + A_6^{\gamma\gamma'}(ncnnn, kkc, knk) + A_6^{\gamma\gamma'}(ncnnn, kkc, nkk) + A_6^{\gamma\gamma'}(ncnnn, kkn, cck) + A_6^{\gamma\gamma'}(ncnnn, kkn, nkk), \quad (A121)$$

$$A_6^{\gamma\gamma'}(ncnnn, kkc, nkk) = A_6^{\gamma\gamma'}(ncnnn, kkc, nkk, c) + A_6^{\gamma\gamma'}(ncnnn, kkc, nkk, n), \quad (A122)$$

$$A_6^{\gamma\gamma'}(ncnnn, kkc, nkk) = A_6^{\gamma\gamma'}(ncnnn, kkc, nkk, ck) + A_6^{\gamma\gamma'}(ncnnn, kkc, nkk, nk), \quad (A123)$$

$$A_6^{\gamma\gamma'}(ncnnn, kkn, cck) = A_6^{\gamma\gamma'}(ncnnn, kkn, cck) + A_6^{\gamma\gamma'}(ncnnn, kkn, ckn), \quad (A124)$$

$$\begin{aligned}
A_6^{\gamma\gamma'}(ncnnn, kknn, nkk) &= A_6^{\gamma\gamma'}(ncnnn, kknn, nkc, ck) + A_6^{\gamma\gamma'}(ncnnn, kknn, nkc, nk) \\
&\quad + A_6^{\gamma\gamma'}(ncnnn, kknn, nkn, ck) + A_6^{\gamma\gamma'}(ncnnn, kknn, nkn, nk, c) \\
&\quad + A_6^{\gamma\gamma'}(ncnnn, kknn, nkn, nk, n)
\end{aligned} \tag{A125}$$

$$\begin{aligned}
A_6^{\gamma\gamma'}(nnenn) &= A_6^{\gamma\gamma'}(nnenn, ckck, kck) + A_6^{\gamma\gamma'}(nnenn, ckck, knk) \\
&\quad + A_6^{\gamma\gamma'}(nnenn, ckkn, kck) + A_6^{\gamma\gamma'}(nnenn, nkck, kck) \\
&\quad + A_6^{\gamma\gamma'}(nnenn, ckkn, knk) + A_6^{\gamma\gamma'}(nnenn, nkck, knk) \\
&\quad + A_6^{\gamma\gamma'}(nnenn, nkkn, kck) + A_6^{\gamma\gamma'}(nnenn, nkkn, knk),
\end{aligned} \tag{A126}$$

$$A_6^{\gamma\gamma'}(nnenn, ckkn, knk) = A_6^{\gamma\gamma'}(nnenn, ckkn, knk, kc) + A_6^{\gamma\gamma'}(nnenn, ckkn, knk, kn), \tag{A127}$$

$$A_6^{\gamma\gamma'}(nnenn, nkck, knk) = A_6^{\gamma\gamma'}(nnenn, nkck, knk, ck) + A_6^{\gamma\gamma'}(nnenn, nkck, knk, nk), \tag{A128}$$

$$A_6^{\gamma\gamma'}(nnenn, nkkn, kck) = A_6^{\gamma\gamma'}(nnenn, nkkn, kck, c) + A_6^{\gamma\gamma'}(nnenn, nkkn, kck, n), \tag{A129}$$

$$\begin{aligned}
A_6^{\gamma\gamma'}(nnenn, nkkn, knk) &= A_6^{\gamma\gamma'}(nnenn, nkkn, knk, cc) + A_6^{\gamma\gamma'}(nnenn, nkkn, knk, cn) \\
&\quad + A_6^{\gamma\gamma'}(nnenn, nkkn, knk, nc) + A_6^{\gamma\gamma'}(nnenn, nkkn, knk, nn, c) \\
&\quad + A_6^{\gamma\gamma'}(nnenn, nkkn, knk, nn, n)
\end{aligned} \tag{A130}$$

$$\begin{aligned}
A_6^{\gamma\gamma'}(nnnen) &= A_6^{\gamma\gamma'}(nnnen, cckk, kkc) + A_6^{\gamma\gamma'}(nnnen, cckk, knk) \\
&\quad + A_6^{\gamma\gamma'}(nnnen, cnkk, kkc) + A_6^{\gamma\gamma'}(nnnen, nckk, kkc) \\
&\quad + A_6^{\gamma\gamma'}(nnnen, cnkk, knk) + A_6^{\gamma\gamma'}(nnnen, nckk, knk) \\
&\quad + A_6^{\gamma\gamma'}(nnnen, nnkk, kkc) + A_6^{\gamma\gamma'}(nnnen, nnkk, knk),
\end{aligned} \tag{A131}$$

$$A_6^{\gamma\gamma'}(nnnen, cnkk, knk) = A_6^{\gamma\gamma'}(nnnen, cnkk, knk, kc) + A_6^{\gamma\gamma'}(nnnen, cnkk, knk, kn), \tag{A132}$$

$$A_6^{\gamma\gamma'}(nnnen, nckk, knk) = A_6^{\gamma\gamma'}(nnnen, nckk, knk, c) + A_6^{\gamma\gamma'}(nnnen, nckk, knk, n), \tag{A133}$$

$$A_6^{\gamma\gamma'}(nnnen, nnkk, kkc) = A_6^{\gamma\gamma'}(nnnen, nnkk, ckc) + A_6^{\gamma\gamma'}(nnnen, nnkk, nkc), \tag{A134}$$

$$\begin{aligned}
A_6^{\gamma\gamma'}(nnnen, nnkk, knk) &= A_6^{\gamma\gamma'}(nnnen, nnkk, ckn, kc) + A_6^{\gamma\gamma'}(nnnen, nnkk, ckn, kn) \\
&\quad + A_6^{\gamma\gamma'}(nnnen, nnkk, nkn, kc) + A_6^{\gamma\gamma'}(nnnen, nnkk, nkn, kn, c) \\
&\quad + A_6^{\gamma\gamma'}(nnnen, nnkk, nkn, kn, n).
\end{aligned} \tag{A135}$$

$$\begin{aligned}
A_6^{\gamma\gamma'}(nnnnnc) &= A_6^{\gamma\gamma'}(nnnnnc, ccck) + A_6^{\gamma\gamma'}(nnnnnc, ccnk) + A_6^{\gamma\gamma'}(nnnnnc, cnck) \\
&\quad + A_6^{\gamma\gamma'}(nnnnnc, nckk) + A_6^{\gamma\gamma'}(nnnnnc, cknk) + A_6^{\gamma\gamma'}(nnnnnc, ncnk) \\
&\quad + A_6^{\gamma\gamma'}(nnnnnc, nnck) + A_6^{\gamma\gamma'}(nnnnnc, nnnk),
\end{aligned} \tag{A136}$$

$$A_6^{\gamma\gamma'}(nnnnnc, cknk) = A_6^{\gamma\gamma'}(nnnnnc, cknk, kck) + A_6^{\gamma\gamma'}(nnnnnc, cknk, knk), \tag{A137}$$

$$A_6^{\gamma\gamma'}(nnnnnc, ncnk) = A_6^{\gamma\gamma'}(nnnnnc, ncnk, ck) + A_6^{\gamma\gamma'}(nnnnnc, ncnk, nk), \tag{A138}$$

$$A_6^{\gamma\gamma'}(nnnnnc, nnck) = A_6^{\gamma\gamma'}(nnnnnc, nnck, c) + A_6^{\gamma\gamma'}(nnnnnc, nnck, n), \tag{A139}$$

$$\begin{aligned}
A_6^{\gamma\gamma'}(nnnnnc, nnnk) &= A_6^{\gamma\gamma'}(nnnnnc, nnnk, cck) + A_6^{\gamma\gamma'}(nnnnnc, nnnk, cnk) \\
&\quad + A_6^{\gamma\gamma'}(nnnnnc, nnnk, nck) + A_6^{\gamma\gamma'}(nnnnnc, nnnk, nnk, ck) \\
&\quad + A_6^{\gamma\gamma'}(nnnnnc, nnnk, nnk, nk).
\end{aligned} \tag{A140}$$

$$\begin{aligned}
A_6^{\gamma\gamma'}(nnnnn) = & A_6^{\gamma\gamma'}(nnnnn, cccc) + A_6^{\gamma\gamma'}(nnnnn, ccen) + A_6^{\gamma\gamma'}(nnnnn, ccnc) \\
& + A_6^{\gamma\gamma'}(nnnnn, cncc) + A_6^{\gamma\gamma'}(nnnnn, nccc) + A_6^{\gamma\gamma'}(nnnnn, ccnn) \\
& + A_6^{\gamma\gamma'}(nnnnn, cncn) + A_6^{\gamma\gamma'}(nnnnn, cnnc) + A_6^{\gamma\gamma'}(nnnnn, nccn) \\
& + A_6^{\gamma\gamma'}(nnnnn, ncnc) + A_6^{\gamma\gamma'}(nnnnn, nncc) + A_6^{\gamma\gamma'}(nnnnn, cnnn) \\
& + A_6^{\gamma\gamma'}(nnnnn, ncnn) + A_6^{\gamma\gamma'}(nnnnn, nncn) + A_6^{\gamma\gamma'}(nnnnn, nnnc) \\
& + A_6^{\gamma\gamma'}(nnnnn, nnnn),
\end{aligned} \tag{A141}$$

$$A_6^{\gamma\gamma'}(nnnnn, ccnn) = A_6^{\gamma\gamma'}(nnnnn, ccnn, kkc) + A_6^{\gamma\gamma'}(nnnnn, ccnn, kn), \tag{A142}$$

$$A_6^{\gamma\gamma'}(nnnnn, cncn) = A_6^{\gamma\gamma'}(nnnnn, cncn, kc) + A_6^{\gamma\gamma'}(nnnnn, cncn, kn), \tag{A143}$$

$$A_6^{\gamma\gamma'}(nnnnn, cnnc) = A_6^{\gamma\gamma'}(nnnnn, cnnc, kck) + A_6^{\gamma\gamma'}(nnnnn, cnnc, knk), \tag{A144}$$

$$A_6^{\gamma\gamma'}(nnnnn, nccn) = A_6^{\gamma\gamma'}(nnnnn, nccn, c) + A_6^{\gamma\gamma'}(nnnnn, nccn, n), \tag{A145}$$

$$A_6^{\gamma\gamma'}(nnnnn, ncnc) = A_6^{\gamma\gamma'}(nnnnn, ncnc, ck) + A_6^{\gamma\gamma'}(nnnnn, ncnc, nk), \tag{A146}$$

$$A_6^{\gamma\gamma'}(nnnnn, nncc) = A_6^{\gamma\gamma'}(nnnnn, nncc, ckk) + A_6^{\gamma\gamma'}(nnnnn, nncc, nkk), \tag{A147}$$

$$\begin{aligned}
A_6^{\gamma\gamma'}(nnnnn, cnnn) = & A_6^{\gamma\gamma'}(nnnnn, cnnn, kcc) + A_6^{\gamma\gamma'}(nnnnn, cnnn, kcn) \\
& + A_6^{\gamma\gamma'}(nnnnn, cnnn, kn) + A_6^{\gamma\gamma'}(nnnnn, cnnn, knn, kc) \\
& + A_6^{\gamma\gamma'}(nnnnn, cnnn, knn, kn),
\end{aligned} \tag{A148}$$

$$\begin{aligned}
A_6^{\gamma\gamma'}(nnnnn, ncnn) = & A_6^{\gamma\gamma'}(nnnnn, ncnn, kkc, ck) + A_6^{\gamma\gamma'}(nnnnn, ncnn, kkc, nk) \\
& + A_6^{\gamma\gamma'}(nnnnn, ncnn, kkn, ck) + A_6^{\gamma\gamma'}(nnnnn, ncnn, kkn, nk, c) \\
& + A_6^{\gamma\gamma'}(nnnnn, ncnn, kkn, nk, n),
\end{aligned} \tag{A149}$$

$$\begin{aligned}
A_6^{\gamma\gamma'}(nnnnn, nncn) = & A_6^{\gamma\gamma'}(nnnnn, nncn, ckk, kc) + A_6^{\gamma\gamma'}(nnnnn, nncn, ckk, kn) \\
& + A_6^{\gamma\gamma'}(nnnnn, nncn, nkk, kc) + A_6^{\gamma\gamma'}(nnnnn, nncn, nkk, kn, c) \\
& + A_6^{\gamma\gamma'}(nnnnn, nncn, nkk, kn, n),
\end{aligned} \tag{A150}$$

$$\begin{aligned}
A_6^{\gamma\gamma'}(nnnnn, nnnc) = & A_6^{\gamma\gamma'}(nnnnn, nnnc, cck) + A_6^{\gamma\gamma'}(nnnnn, nnnc, cnk) \\
& + A_6^{\gamma\gamma'}(nnnnn, nnnc, nck) + A_6^{\gamma\gamma'}(nnnnn, nnnc, nnk, ck) \\
& + A_6^{\gamma\gamma'}(nnnnn, nnnc, nnk, nk),
\end{aligned} \tag{A151}$$

$$\begin{aligned}
A_6^{\gamma\gamma'}(nnnnn, nnnn) = & A_6^{\gamma\gamma'}(nnnnn, nnnn, ccc) + A_6^{\gamma\gamma'}(nnnnn, nnnn, ccn) \\
& + A_6^{\gamma\gamma'}(nnnnn, nnnn, cnc) + A_6^{\gamma\gamma'}(nnnnn, nnnn, ncc) \\
& + A_6^{\gamma\gamma'}(nnnnn, nnnn, cnn) + A_6^{\gamma\gamma'}(nnnnn, nnnn, ncn) \\
& + A_6^{\gamma\gamma'}(nnnnn, nnnn, nnc) + A_6^{\gamma\gamma'}(nnnnn, nnnn, nnn),
\end{aligned} \tag{A152}$$

$$A_6^{\gamma\gamma'}(nnnnn, nnnn, cnn) = A_6^{\gamma\gamma'}(nnnnn, nnnn, cnn, kc) + A_6^{\gamma\gamma'}(nnnnn, nnnn, cnn, kn), \tag{A153}$$

$$A_6^{\gamma\gamma'}(nnnnn, nnnn, ncn) = A_6^{\gamma\gamma'}(nnnnn, nnnn, ncn, c) + A_6^{\gamma\gamma'}(nnnnn, nnnn, ncn, n), \tag{A154}$$

$$A_6^{\gamma\gamma'}(nnnnn, nnnn, nnc) = A_6^{\gamma\gamma'}(nnnnn, nnnn, nnc, ck) + A_6^{\gamma\gamma'}(nnnnn, nnnn, nnc, nk), \tag{A155}$$

$$\begin{aligned}
A_6^{\gamma\gamma'}(nnnnn, nnnn, nnn) = & A_6^{\gamma\gamma'}(nnnnn, nnnn, nnn, cc) + A_6^{\gamma\gamma'}(nnnnn, nnnn, nnn, cn) \\
& + A_6^{\gamma\gamma'}(nnnnn, nnnn, nnn, nc) + A_6^{\gamma\gamma'}(nnnnn, nnnn, nnn, nn, c) \\
& + A_6^{\gamma\gamma'}(nnnnn, nnnn, nnn, nn, n).
\end{aligned} \tag{A156}$$

Thus, we obtain that the contribution from the six order approximation is made of 203 terms after finding out all of contractions and anti-contractions.

Just like we have done in the $l = 4$ or 5 cases, we decompose

$$A_6^{\gamma\gamma'} = A_6^{\gamma\gamma'}(e) + A_6^{\gamma\gamma'}(te) + A_6^{\gamma\gamma'}(t^2e) + A_6^{\gamma\gamma'}(t^3e) \quad (\text{A157})$$

$$= A_6^{\gamma\gamma'}(e, te) + A_6^{\gamma\gamma'}(t^2e, t^3e), \quad (\text{A158})$$

To our purpose, we only calculate the second term $A_6^{\gamma\gamma'}(t^2e, t^3e)$ in eq.(A158). Without loss of generality, we decompose it into

$$\begin{aligned} A_6^{\gamma\gamma'}(t^2e, t^3e) &= A_6^{\gamma\gamma'}(t^2e^{-iE_\gamma t}, t^3e^{-iE_\gamma t}) + A_6^{\gamma\gamma'}(t^2e^{-iE_{\gamma_1} t}, t^3e^{-iE_{\gamma_1} t}) \\ &\quad + A_6^{\gamma\gamma'}(t^2e^{-iE_{\gamma'} t}, t^3e^{-iE_{\gamma'} t}). \end{aligned} \quad (\text{A159})$$

While, every term in the above equations has its diagonal and off-diagonal parts about γ and γ' , that is

$$A_6^{\gamma\gamma'}(t^2e^{-iE_{\gamma_i} t}) = A_6^{\gamma\gamma'}(t^2e^{-iE_{\gamma_i} t}; D) + A_6^{\gamma\gamma'}(t^2e^{-iE_{\gamma_i} t}; N), \quad (\text{A160})$$

$$A_6^{\gamma\gamma'}(t^3e^{-iE_{\gamma_i} t}) = A_6^{\gamma\gamma'}(t^3e^{-iE_{\gamma_i} t}; D) + A_6^{\gamma\gamma'}(t^3e^{-iE_{\gamma_i} t}; N). \quad (\text{A161})$$

where E_{γ_i} takes E_γ, E_{γ_1} and $E_{\gamma'}$.

Based on our calculations, we find that there are nonvanishing 91 terms and vanishing 112 terms with t^2e, t^3e factor parts in all of 203 contraction- and anti contraction- expressions (see in the supplementary Ref. [10]). Therefore we can, in terms of rearrangement and summation, obtain the following concise forms:

$$\begin{aligned} A_6(t^2e^{-iE_\gamma t}, D) &= \frac{(-it)^2}{2!} \left(G_\gamma^{(3)}\right)^2 e^{-iE_\gamma t} \delta_{\gamma\gamma'} + \frac{(-it)^2}{2!} 2G_\gamma^{(2)} G_\gamma^{(4)} e^{-iE_\gamma t} \delta_{\gamma\gamma'} \\ &\quad - \frac{(-it)^2}{2!} \sum_{\gamma_1} \frac{\left(G_\gamma^{(2)}\right)^2 e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma} \delta_{\gamma\gamma'}. \end{aligned} \quad (\text{A162})$$

$$A_6(t^2e^{-iE_{\gamma_1} t}, D) = \frac{(-it)^2}{2!} \sum_{\gamma_1} \frac{\left(G_\gamma^{(2)}\right)^2 e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^2} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma} \delta_{\gamma\gamma'}. \quad (\text{A163})$$

$$A_6(t^2e^{-iE_{\gamma'} t}, D) = 0. \quad (\text{A164})$$

$$\begin{aligned} A_6(t^2e^{-iE_\gamma t}, N) &= \frac{(-it)^2}{2!} 2G_\gamma^{(2)} G_\gamma^{(3)} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma'})} g_1^{\gamma\gamma'} \\ &\quad + \frac{(-it)^2}{2!} \sum_{\gamma_1} \frac{\left(G_\gamma^{(2)}\right)^2 e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma'})} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'}. \end{aligned} \quad (\text{A165})$$

$$A_6(t^2e^{-iE_{\gamma_1} t}, N) = -\frac{(-it)^2}{2!} \sum_{\gamma_1} \frac{\left(G_{\gamma_1}^{(2)}\right)^2 e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma'})} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'}. \quad (\text{A166})$$

$$\begin{aligned} A_6(t^2e^{-iE_{\gamma'} t}, N) &= -\frac{(-it)^2}{2!} 2G_{\gamma'}^{(2)} G_{\gamma'}^{(3)} \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})} g_1^{\gamma\gamma'} \\ &\quad + \frac{(-it)^2}{2!} \sum_{\gamma_1} \frac{\left(G_{\gamma'}^{(2)}\right)^2 e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'})} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'}. \end{aligned} \quad (\text{A167})$$

Their forms are indeed the same as expected and can be absorbed reasonably into the lower order approximations in order to obtain the improved forms of perturbed solutions.

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SUPPLEMENT

1 $l = 5$ case

In the following, we respectively calculate the 52 component expressions of $A_5^{\gamma\gamma'}$, and put the second and third terms in eq.(A68) together as $A_5^{\gamma\gamma'}(te, t^2e)$.

$$A_5^{\gamma\gamma'}(cccc; te, t^2e) = \left[-(-it) \frac{3e^{-iE_\gamma t} + 3e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^4} + \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma'})^3} - \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^3} \right] |g_1^{\gamma\gamma'}|^4 g_1^{\gamma\gamma'}. \quad (168)$$

$$A_5^{\gamma\gamma'}(cccn; te, t^2e) = \sum_{\gamma_1} \left[-(-it) \frac{2e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^3 (E_\gamma - E_{\gamma'})} - (-it) \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma'})^2} - (-it) \frac{e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^3 (E_{\gamma_1} - E_{\gamma'})} + \frac{(-it)^2}{2!} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma'})} \right] \times |g_1^{\gamma\gamma_1}|^4 g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma'}. \quad (169)$$

$$A_5^{\gamma\gamma'}(ccnc; te, t^2e) = \sum_{\gamma_1} \left[(-it) \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma'})^3} - (-it) \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})^2} - (-it) \frac{2e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^3 (E_{\gamma_1} - E_{\gamma'})} - \frac{(-it)^2}{2!} \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})} \right] \times |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma'}. \quad (170)$$

$$A_5^{\gamma\gamma'}(cncc; te, t^2e) = \sum_{\gamma_1} \left[-(-it) \frac{2e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma'})^3} - (-it) \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma'})^2} + (-it) \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^3 (E_{\gamma_1} - E_{\gamma'})} + \frac{(-it)^2}{2!} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma'})^2} \right] \times |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma'}. \quad (171)$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(nccc; te, t^2e) = & \sum_{\gamma_1} \left[(-it) \frac{e^{-iE_{\gamma_1}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma'})^3} - (-it) \frac{2e^{-iE_{\gamma'}t}}{(E_{\gamma} - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})^3} \right. \\
& \left. - (-it) \frac{e^{-iE_{\gamma'}t}}{(E_{\gamma} - E_{\gamma'})^2(E_{\gamma_1} - E_{\gamma'})^2} - \frac{(-it)^2}{2!} \frac{e^{-iE_{\gamma'}t}}{(E_{\gamma} - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})^2} \right] \\
& \times |g_1^{\gamma_1\gamma'}|^4 g_1^{\gamma\gamma'} \eta_{\gamma\gamma_1}.
\end{aligned} \tag{172}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(ccnn, kkc; te, t^2e) = & \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})^2(E_{\gamma} - E_{\gamma_2})^2} - (-it) \frac{2e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})^3(E_{\gamma} - E_{\gamma_2})} \right. \\
& \left. - (-it) \frac{e^{-iE_{\gamma_1}t}}{(E_{\gamma} - E_{\gamma_1})^3(E_{\gamma_1} - E_{\gamma_2})} + \frac{(-it)^2}{2!} \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})^2(E_{\gamma} - E_{\gamma_2})} \right] \\
& \times |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} \delta_{\gamma\gamma'}.
\end{aligned} \tag{173}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(ccnn, kkn; te, t^2e) = & \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})^2(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma'})} \right. \\
& \left. + (-it) \frac{e^{-iE_{\gamma_1}t}}{(E_{\gamma} - E_{\gamma_1})^2(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma'})} \right] \\
& \times |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma_1\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma'}.
\end{aligned} \tag{174}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(cncn, kc; te, t^2e) = & \sum_{\gamma_1} \left[(-it) \frac{2e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma'})^3} - (-it) \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})^2(E_{\gamma} - E_{\gamma'})^2} \right. \\
& \left. + (-it) \frac{e^{-iE_{\gamma'}t}}{(E_{\gamma} - E_{\gamma'})^3(E_{\gamma_1} - E_{\gamma'})} + \frac{(-it)^2}{2!} \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma'})^2} \right] \\
& \times |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma'}.
\end{aligned} \tag{175}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(cncn, kn; te, t^2e) = & \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma'})^2} \right. \\
& - (-it) \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})^2(E_{\gamma} - E_{\gamma'})} \\
& - (-it) \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})^2(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma'})} \\
& \left. + \frac{(-it)^2}{2!} \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma'})} \right] \\
& \times |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma_2}|^2 g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma_2} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma'}.
\end{aligned} \tag{176}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(cnnc, kck; te, t^2e) = & \sum_{\gamma_1} \left[(-it) \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})^2(E_{\gamma} - E_{\gamma'})^2} + (-it) \frac{e^{-iE_{\gamma_1}t}}{(E_{\gamma} - E_{\gamma_1})^2(E_{\gamma_1} - E_{\gamma'})^2} \right. \\
& \left. + (-it) \frac{e^{-iE_{\gamma'}t}}{(E_{\gamma} - E_{\gamma'})^2(E_{\gamma_1} - E_{\gamma'})^2} \right] |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma_1\gamma'}|^2 g_1^{\gamma\gamma'}.
\end{aligned} \tag{177}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(cnnc, knk; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})^2} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2(E_{\gamma_1} - E_{\gamma'})^2(E_{\gamma_2} - E_{\gamma'})} \right] \\
&\quad \times |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma_2\gamma'}|^2 g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma'} \eta_{\gamma_1\gamma_2} \eta_{\gamma\gamma_2}.
\end{aligned} \tag{178}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(nccn, c; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2(E_\gamma - E_{\gamma_2})^2} + (-it) \frac{e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^2(E_{\gamma_1} - E_{\gamma_2})^2} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma_2} t}}{(E_\gamma - E_{\gamma_2})^2(E_{\gamma_1} - E_{\gamma_2})^2} \right] |g_1^{\gamma_1\gamma_2}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} \delta_{\gamma\gamma'}.
\end{aligned} \tag{179}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(nccn, n; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[-(-it) \frac{e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma_2})^2(E_{\gamma_1} - E_{\gamma'})} \right. \\
&\quad \left. - (-it) \frac{e^{-iE_{\gamma_2} t}}{(E_\gamma - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma_2})^2(E_{\gamma_2} - E_{\gamma'})} \right] \\
&\quad \times |g_1^{\gamma_1\gamma_2}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma_1\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma'}.
\end{aligned} \tag{180}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(ncnc, ck; te, t^2e) &= \sum_{\gamma_1} \left[(-it) \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma'})^3} - (-it) \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2(E_{\gamma_1} - E_{\gamma'})^2} \right. \\
&\quad \left. - (-it) \frac{2e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^3(E_{\gamma_1} - E_{\gamma'})} - \frac{(-it)^2}{2!} \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2(E_{\gamma_1} - E_{\gamma'})} \right] \\
&\quad \times |g_1^{\gamma_1\gamma'}|^2 |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma'} \eta_{\gamma\gamma_1}.
\end{aligned} \tag{181}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(ncnc, nk; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[-(-it) \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})(E_{\gamma_2} - E_{\gamma'})^2} \right. \\
&\quad - (-it) \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})^2(E_{\gamma_2} - E_{\gamma'})} \\
&\quad - (-it) \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2(E_{\gamma_1} - E_{\gamma'})(E_{\gamma_2} - E_{\gamma'})} \\
&\quad \left. - \frac{(-it)^2}{2!} \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})(E_{\gamma_2} - E_{\gamma'})} \right] \\
&\quad \times |g_1^{\gamma_1\gamma'}|^2 |g_1^{\gamma_2\gamma'}|^2 g_1^{\gamma\gamma'} \eta_{\gamma\gamma_1} \eta_{\gamma\gamma_2} \eta_{\gamma_1\gamma_2}.
\end{aligned} \tag{182}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(nncc, ckk; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[-(-it) \frac{2e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})^3} - (-it) \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2(E_\gamma - E_{\gamma_2})^2} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma_2} t}}{(E_\gamma - E_{\gamma_2})^3(E_{\gamma_1} - E_{\gamma_2})} + \frac{(-it)^2}{2!} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})^2} \right] \\
&\quad \times |g_1^{\gamma\gamma_2}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} \delta_{\gamma\gamma'}.
\end{aligned} \tag{183}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(nncc, nkk; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_{\gamma_2}t}}{(E_{\gamma} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_2} - E_{\gamma'})^2} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma'}t}}{(E_{\gamma} - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})(E_{\gamma_2} - E_{\gamma'})^2} \right] \\
&\quad \times |g_1^{\gamma_2\gamma'}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma_1\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma'}.
\end{aligned} \tag{184}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(cnnn, kcc; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{2e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})^3 (E_{\gamma} - E_{\gamma_2})} - (-it) \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma_2})^2} \right. \\
&\quad \left. - (-it) \frac{e^{-iE_{\gamma_1}t}}{(E_{\gamma} - E_{\gamma_1})^3 (E_{\gamma_1} - E_{\gamma_2})} + \frac{(-it)^2}{2!} \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma_2})} \right] \\
&\quad \times |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma_1} g_1^{\gamma_1\gamma} \delta_{\gamma\gamma'}.
\end{aligned} \tag{185}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(cnnn, kcn; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma'})} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma_1}t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma'})} \right] \\
&\quad \times |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma_2\gamma'} \eta_{\gamma\gamma'}.
\end{aligned} \tag{186}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(cnnn, knc; te, t^2e) &= \sum_{\gamma_1, \gamma_2, \gamma_3} \left[(-it) \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma_3})} \right. \\
&\quad - (-it) \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})^2 (E_{\gamma} - E_{\gamma_3})} \\
&\quad - (-it) \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma_3})^2} \\
&\quad \left. + \frac{(-it)^2}{2!} \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma_3})} \right] \\
&\quad \times |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma} \eta_{\gamma_1\gamma_2} \eta_{\gamma_1\gamma_3} \delta_{\gamma\gamma'}.
\end{aligned} \tag{187}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(cnnn, knn, kc; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma'})^2} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma'}t}}{(E_{\gamma} - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})} \right] \\
&\quad \times |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma_1\gamma'}.
\end{aligned} \tag{188}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(cnnn, knn, kn; te, t^2e) &= \sum_{\gamma_1, \gamma_2, \gamma_3} (-it) \frac{e^{-iE_{\gamma}t} |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma'} \eta_{\gamma\gamma_3} \eta_{\gamma\gamma'} \eta_{\gamma_1\gamma_2} \eta_{\gamma_1\gamma_3} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma'}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma_3})(E_{\gamma} - E_{\gamma'})}.
\end{aligned} \tag{189}$$

$$\begin{aligned}
& A_5^{\gamma\gamma'}(ncnn, kkc, ck; te, t^2e) \\
&= \sum_{\gamma_1} \left[(-it) \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma'})^2} + (-it) \frac{e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma'})^2} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})^2} \right] |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma_1\gamma'}|^2 g_1^{\gamma\gamma'}.
\end{aligned} \tag{190}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(ncnn, kkc, nk; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1}) (E_{\gamma_1} - E_{\gamma_2}) (E_{\gamma_1} - E_{\gamma'})^2} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'})^2 (E_{\gamma_2} - E_{\gamma'})} \right] \\
&\quad \times |g_1^{\gamma_1\gamma'}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma'}.
\end{aligned} \tag{191}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(ncnn, kkn, ck; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma'})} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2}) (E_{\gamma_1} - E_{\gamma'})} \right] \\
&\quad \times |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma_1\gamma_2}|^2 g_1^{\gamma\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma'}.
\end{aligned} \tag{192}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(ncnn, kkn, nk, c; te, t^2e) &= \sum_{\gamma_1, \gamma_2, \gamma_3} \left[(-it) \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma_3})} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2}) (E_{\gamma_1} - E_{\gamma_3})} \right] \\
&\quad \times |g_1^{\gamma_1\gamma_2}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_3} g_1^{\gamma_3\gamma} \eta_{\gamma\gamma_2} \eta_{\gamma_2\gamma_3} \delta_{\gamma\gamma'}.
\end{aligned} \tag{193}$$

$$\begin{aligned}
& A_5^{\gamma\gamma'}(ncnn, kkn, nk, n; te, t^2e) \\
&= - \sum_{\gamma_1, \gamma_2, \gamma_3} (-it) \frac{e^{-iE_{\gamma_1} t} |g_1^{\gamma_1\gamma_2}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_3} g_1^{\gamma_3\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma_3} \eta_{\gamma\gamma'} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma_3} \eta_{\gamma_2\gamma'}}{(E_\gamma - E_{\gamma_1}) (E_{\gamma_1} - E_{\gamma_2}) (E_{\gamma_1} - E_{\gamma_3}) (E_{\gamma_1} - E_{\gamma'})}.
\end{aligned} \tag{194}$$

$$\begin{aligned}
& A_5^{\gamma\gamma'}(nncn, ckk; kc, te, t^2e) \\
&= \sum_{\gamma_1} \left[(-it) \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma'})^2} + (-it) \frac{e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma'})^2} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})^2} \right] |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma_1\gamma'}|^2 g_1^{\gamma\gamma'}.
\end{aligned} \tag{195}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(nncn, ckk, kn; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2})^2 (E_\gamma - E_{\gamma'})} \right. \\
&\quad \left. - (-it) \frac{e^{-iE_{\gamma_2} t}}{(E_\gamma - E_{\gamma_2})^2 (E_{\gamma_1} - E_{\gamma_2}) (E_{\gamma_2} - E_{\gamma'})} \right] \\
&\quad \times |g_1^{\gamma\gamma_2}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma'} \eta_{\gamma_1\gamma'}.
\end{aligned} \tag{196}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(nncn, nkk, kc; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_{\gamma_1}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma'})^2} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma'}t}}{(E_{\gamma} - E_{\gamma'})^2(E_{\gamma_1} - E_{\gamma'})^2(E_{\gamma_2} - E_{\gamma'})} \right] \\
&\quad \times |g_1^{\gamma_1\gamma_2}|^2 |g_1^{\gamma_1\gamma'}|^2 g_1^{\gamma\gamma'} \eta_{\gamma\gamma_1} \eta_{\gamma\gamma_2} \eta_{\gamma_2\gamma'}.
\end{aligned} \tag{197}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(nncn, nkk, kn, c; te, t^2e) &= \sum_{\gamma_1, \gamma_2, \gamma_3} \left[(-it) \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})^2(E_{\gamma} - E_{\gamma_3})} \right. \\
&\quad \left. - (-it) \frac{e^{-iE_{\gamma_2}t}}{(E_{\gamma} - E_{\gamma_2})^2(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_2} - E_{\gamma_3})} \right] \\
&\quad \times |g_1^{\gamma_2\gamma_3}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} \eta_{\gamma\gamma_3} \eta_{\gamma_1\gamma_3} \delta_{\gamma\gamma'}.
\end{aligned} \tag{198}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(nncn, nkk, kn, n; te, t^2e) \\
&= \sum_{\gamma_1, \gamma_2, \gamma_3} (-it) \frac{e^{-iE_{\gamma_2}t} |g_1^{\gamma_2\gamma_3}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma_3} \eta_{\gamma\gamma'} \eta_{\gamma_1\gamma_3} \eta_{\gamma_1\gamma'} \eta_{\gamma_3\gamma'}}{(E_{\gamma} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_2} - E_{\gamma_3})(E_{\gamma_2} - E_{\gamma'})}.
\end{aligned} \tag{199}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(nnnc, cck; te, t^2e) \\
&= \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})^2(E_{\gamma} - E_{\gamma_2})^2} - (-it) \frac{2e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})^3(E_{\gamma} - E_{\gamma_2})} \right. \\
&\quad \left. - (-it) \frac{e^{-iE_{\gamma_1}t}}{(E_{\gamma} - E_{\gamma_1})^3(E_{\gamma_1} - E_{\gamma_2})} + \frac{(-it)^2}{2!} \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})^2(E_{\gamma} - E_{\gamma_2})} \right] \\
&\quad \times |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} \delta_{\gamma\gamma'}.
\end{aligned} \tag{200}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(nnnc, cnk; te, t^2e) &= \sum_{\gamma_1, \gamma_2, \gamma_3} \left[(-it) \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma_3})^2} \right. \\
&\quad - (-it) \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})^2(E_{\gamma} - E_{\gamma_3})} \\
&\quad - (-it) \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})^2(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma_3})} \\
&\quad \left. + \frac{(-it)^2}{2!} \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma_3})} \right] \\
&\quad \times |g_1^{\gamma\gamma_3}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_3} \eta_{\gamma_2\gamma_3} \delta_{\gamma\gamma'}.
\end{aligned} \tag{201}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(nnnc, nck; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_{\gamma_1}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma'})^2} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma'}t}}{(E_{\gamma} - E_{\gamma'})^2(E_{\gamma_1} - E_{\gamma'})^2(E_{\gamma_2} - E_{\gamma'})} \right] \\
&\quad \times |g_1^{\gamma_1\gamma'}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma'}.
\end{aligned} \tag{202}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(nnnc, nnk, ck; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})^2} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})} \right] \\
&\quad \times \left| g_1^{\gamma\gamma'} \right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma_1\gamma'}.
\end{aligned} \tag{203}$$

$$\begin{aligned}
&A_5^{\gamma\gamma'}(nnnc, nnk, nk; te, t^2e) \\
&= \sum_{\gamma_1, \gamma_2, \gamma_3} (-it) \frac{e^{-iE_{\gamma'} t} \left| g_1^{\gamma_3\gamma'} \right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma_3} \eta_{\gamma\gamma'} \eta_{\gamma_1\gamma_3} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma_3}}{(E_\gamma - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'}) (E_{\gamma_3} - E_{\gamma'})}.
\end{aligned} \tag{204}$$

$$\begin{aligned}
&A_5^{\gamma\gamma'}(nnnn, ccc; te, t^2e) \\
&= \sum_{\gamma_1} \left[(-it) \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma'})^2} + (-it) \frac{e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma'})^2} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})^2} \right] \left| g_1^{\gamma\gamma_1} \right|^2 \left(g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \right)^2 g_1^{\gamma\gamma'}.
\end{aligned} \tag{205}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(nnnn, ccn; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma'})} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2}) (E_{\gamma_1} - E_{\gamma'})} \right] \\
&\quad \times (g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma}) g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'} \eta_{\gamma_2\gamma'}.
\end{aligned} \tag{206}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(nnnn, cnc; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma'})^2} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})} \right] \\
&\quad \times \left(g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \right) \left(g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma'} \right) g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma_2}.
\end{aligned} \tag{207}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(nnnn, ncc; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1}) (E_{\gamma_1} - E_{\gamma_2}) (E_{\gamma_1} - E_{\gamma'})^2} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'})^2 (E_{\gamma_2} - E_{\gamma'})} \right] \\
&\quad \times \left(g_1^{\gamma'\gamma_2} g_1^{\gamma_2\gamma_1} g_1^{\gamma_1\gamma'} \right) g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma'}.
\end{aligned} \tag{208}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(nnnn, cnn, kc; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma'})^2} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})} \right] \\
&\quad \times \left(g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma} \right) \left(g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma'} \right) g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma_2}.
\end{aligned} \tag{209}$$

$$\begin{aligned}
& A_5^{\gamma\gamma'}(nnnn, cn, kn; te, t^2e) \\
&= \sum_{\gamma_1, \gamma_2, \gamma_3} (-it) \frac{e^{-iE_\gamma t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma'} \eta_{\gamma\gamma'} \eta_{\gamma_1\gamma_3} \eta_{\gamma_2\gamma_3} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma'}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})(E_\gamma - E_{\gamma'})}.
\end{aligned} \tag{210}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(nnnn, ncn, c; te, t^2e) &= \sum_{\gamma_1, \gamma_2, \gamma_3} \left[(-it) \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma_3})} \right] \\
&\quad \times |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma_1} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma_3} \delta_{\gamma\gamma'}.
\end{aligned} \tag{211}$$

$$\begin{aligned}
& A_5^{\gamma\gamma'}(nnnn, ncn, n; te, t^2e) \\
&= - \sum_{\gamma_1, \gamma_2, \gamma_3} (-it) \frac{e^{-iE_{\gamma_1} t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma_3} \eta_{\gamma_2\gamma'} \eta_{\gamma_3\gamma'}}{(E_\gamma - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma_3})(E_{\gamma_1} - E_{\gamma'})}.
\end{aligned} \tag{212}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(nnnn, nnc, ck; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})^2} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})} \right] \\
&\quad \times (g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}) (g_1^{\gamma'\gamma_2} g_1^{\gamma_2\gamma}) g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma_2}.
\end{aligned} \tag{213}$$

$$\begin{aligned}
& A_5^{\gamma\gamma'}(nnnn, nnc, nk; te, t^2e) \\
&= \sum_{\gamma_1, \gamma_2, \gamma_3} (-it) \frac{e^{-iE_{\gamma'} t} g_1^{\gamma'\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma'} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma_3} \eta_{\gamma_1\gamma_2} \eta_{\gamma_1\gamma_3}}{(E_\gamma - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'}) (E_{\gamma_3} - E_{\gamma'})}.
\end{aligned} \tag{214}$$

$$\begin{aligned}
A_5^{\gamma\gamma'}(nnnn, nnn, cc; te, t^2e) &= \sum_{\gamma_1, \gamma_2} \left[(-it) \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})^2} \right. \\
&\quad \left. + (-it) \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})} \right] \\
&\quad \times g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma'} \eta_{\gamma\gamma_1} \eta_{\gamma_2\gamma'}.
\end{aligned} \tag{215}$$

$$\begin{aligned}
& A_5^{\gamma\gamma'}(nnnn, nnn, cn; te, t^2e) \\
&= \sum_{\gamma_1, \gamma_2, \gamma_3} (-it) \frac{e^{-iE_\gamma t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma} g_1^{\gamma\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma_1\gamma_3} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma'} \eta_{\gamma_3\gamma'}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})(E_\gamma - E_{\gamma'})}.
\end{aligned} \tag{216}$$

$$\begin{aligned}
& A_5^{\gamma\gamma'}(nnnn, nnn, nc; te, t^2e) \\
&= \sum_{\gamma_1, \gamma_2, \gamma_3} (-it) \frac{e^{-iE_{\gamma'} t} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma'} g_1^{\gamma\gamma'} \eta_{\gamma\gamma_1} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma_3} \eta_{\gamma_1\gamma_3} \eta_{\gamma_2\gamma'}}{(E_\gamma - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'}) (E_{\gamma_3} - E_{\gamma'})}.
\end{aligned} \tag{217}$$

$$\begin{aligned}
& A_5^{\gamma\gamma'}(nnnn, nnn, n, c; te, t^2e) \\
&= \sum_{\gamma_1, \gamma_2, \gamma_3, \gamma_4} (-it) \frac{e^{-iE_\gamma t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma_4} g_1^{\gamma_4\gamma} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma_3} \eta_{\gamma_1\gamma_3} \eta_{\gamma_1\gamma_4} \eta_{\gamma_2\gamma_4} \delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})(E_\gamma - E_{\gamma_4})}.
\end{aligned} \tag{218}$$

$$A_5^{\gamma\gamma'}(nnnn, nnn, nn, n; te, t^2e) = 0 \tag{219}$$

D. $l = 6$ case

Based on our calculations, we find that there are nonvanishing 91 terms and vanishing 112 terms with t^2e, t^3e factor parts in all of 203 contraction- and anti contraction- expressions. In the following, we respectively calculate them term by term, and we only write down the non-zero expressions for saving space.

$$A_6(ccccc; t^2e, t^3e) = \sum_{\gamma_1} \left[\frac{(-it)^3}{3!} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^3} - \frac{(-it)^2}{2} \frac{3e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^4} + \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^4} \right] |g_1^{\gamma\gamma_1}|^6 \delta_{\gamma\gamma'}. \quad (220)$$

$$A_6(ccccn; t^2e, t^3e) = \sum_{\gamma_1} \left[\frac{(-it)^2}{2} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^3 (E_\gamma - E_{\gamma'})} - \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^3 (E_{\gamma_1} - E_{\gamma'})} \right] |g_1^{\gamma\gamma_1}|^4 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'}. \quad (221)$$

$$A_6(cccnc; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \left[\frac{(-it)^3}{3!} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma'})} - \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2})^2} - \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^3 (E_\gamma - E_{\gamma_2})} \right] |g_1^{\gamma\gamma_1}|^4 |g_1^{\gamma\gamma_2}|^2 \eta_{\gamma_1\gamma_2} \delta_{\gamma\gamma'}. \quad (222)$$

$$A_6(ccncc; t^2e, t^3e) = \sum_{\gamma_1} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma'})^2} |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma_1\gamma'}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'}. \quad (223)$$

$$A_6(cnccc; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \left[\frac{(-it)^3}{3!} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2})^2} - \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2})^3} - \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2})^2} \right] |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma_2}|^4 \eta_{\gamma_1\gamma_2} \delta_{\gamma\gamma'}. \quad (224)$$

$$A_6(ncccc; t^2e, t^3e) = \sum_{\gamma_1} \left[-\frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1}) (E_{\gamma_1} - E_{\gamma'})^3} + \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'})^3} \right] |g_1^{\gamma\gamma'}|^4 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'}. \quad (225)$$

$$A_6(cccnn, kkkc; t^2e, t^3e) = \sum_{\gamma_1} \left[\frac{(-it)^2}{2} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma'})^3} + \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^3 (E_{\gamma_1} - E_{\gamma'})} \right] |g_1^{\gamma\gamma'}|^4 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}. \quad (226)$$

$$A_6(cccnn, kkkn; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t} |g_1^{\gamma\gamma_1}|^4 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma'} \eta_{\gamma_1\gamma_2} \eta_{\gamma_1\gamma'}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma'})}. \quad (227)$$

$$A_6(ccncn, kkc; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \left[\frac{(-it)^2}{2} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^3 (E_\gamma - E_{\gamma'})} - \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^3 (E_{\gamma_1} - E_{\gamma_2})} \right] |g_1^{\gamma\gamma_1}|^4 |g_1^{\gamma_1\gamma_2}|^2 \eta_{\gamma\gamma_2} \delta_{\gamma\gamma'}. \quad (228)$$

$$A_6(ccncn, kn; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1} t} |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma_1\gamma_2}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma'} \eta_{\gamma_2\gamma'}}{(E_\gamma - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2}) (E_{\gamma_1} - E_{\gamma'})}. \quad (229)$$

$$A_6(ccnnc, kck; t^2e, t^3e) = \sum_{\gamma_1} \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t} |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma'})^2}. \quad (230)$$

$$A_6(cncnc, kc; t^2e, t^3e) = \sum_{\gamma_1} \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t} |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma'})^2}. \quad (231)$$

$$A_6(cncnc, kn; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t} |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma_2}|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma'} \eta_{\gamma_1\gamma_2} \eta_{\gamma_1\gamma'}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2})^2 (E_\gamma - E_{\gamma'})}. \quad (232)$$

$$A_6(cncnc, kck; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \left[\frac{(-it)^3}{3!} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2})} - \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2})^2} - \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^3 (E_\gamma - E_{\gamma_2})} \right] \times |g_1^{\gamma\gamma_1}|^4 |g_1^{\gamma\gamma_2}|^2 \eta_{\gamma_1\gamma_2} \delta_{\gamma\gamma'}. \quad (233)$$

$$A_6(cncnc, knk; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2, \gamma_3} \left[\frac{(-it)^3}{3!} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma_3})} - \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma_3})^2} - \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2})^2 (E_\gamma - E_{\gamma_3})} - \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma_3})} \right] \times |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma_2}|^2 |g_1^{\gamma\gamma_3}|^2 \eta_{\gamma_1\gamma_2} \eta_{\gamma_1\gamma_3} \eta_{\gamma_2\gamma_3} \delta_{\gamma\gamma'}. \quad (234)$$

$$A_6(cnncc, kck; t^2e, t^3e) = \sum_{\gamma_1} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma'} t} |g_1^{\gamma_1\gamma'}|^2 |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})^2}. \quad (235)$$

$$A_6(ncccn, c; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1} t} |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma_1\gamma_2}|^4 \eta_{\gamma\gamma_2} \delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2})^2}. \quad (236)$$

$$A_6(ncccn, n; t^2e, t^3e) = - \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1}t} |g_1^{\gamma_1\gamma_2}|^4 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma_2\gamma'}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma_2})^2 (E_{\gamma_1} - E_{\gamma'})}. \quad (237)$$

$$A_6(nccnc, ck; t^2e, t^3e) = \sum_{\gamma_1} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma'}t} |g_1^{\gamma\gamma'}|^2 |g_1^{\gamma_2\gamma'}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}}{(E_{\gamma} - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})^2}. \quad (238)$$

$$A_6(nccnc, nk; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma'}t} |g_1^{\gamma_1\gamma'}|^2 |g_1^{\gamma_2\gamma'}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma_1\gamma_2}}{(E_{\gamma} - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'})^2 (E_{\gamma_2} - E_{\gamma'})}. \quad (239)$$

$$A_6(ncncc, ckk; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \left[\frac{(-it)^2}{2} \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})^3 (E_{\gamma} - E_{\gamma_2})} - \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1}t}}{(E_{\gamma} - E_{\gamma_1})^3 (E_{\gamma_1} - E_{\gamma_2})} \right] |g_1^{\gamma\gamma_1}|^4 |g_1^{\gamma_1\gamma_2}|^2 \eta_{\gamma\gamma_2} \delta_{\gamma\gamma'}. \quad (240)$$

$$A_6(ncncc, nkk; t^2e, t^3e) = - \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1}t} |g_1^{\gamma_1\gamma_2}|^2 |g_1^{\gamma_1\gamma'}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma_2\gamma'}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma'})^2}. \quad (241)$$

$$A_6(nnccc, ckkk; t^2e, t^3e) = \sum_{\gamma_1} \left[\frac{(-it)^2}{2} \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma'})^3} + \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma'}t}}{(E_{\gamma} - E_{\gamma'})^3 (E_{\gamma_1} - E_{\gamma'})} \right] |g_1^{\gamma\gamma'}|^4 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}. \quad (242)$$

$$A_6(nnccc, nkkk; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma'}t} |g_1^{\gamma_2\gamma'}|^4 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma_1\gamma_2}}{(E_{\gamma} - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})^2}. \quad (243)$$

$$A_6(ccnnn, kkc; t^2e, t^3e) = \sum_{\gamma_1} \left[\frac{(-it)^2}{2} \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1})(E_{\gamma} - E_{\gamma'})^3} + \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma'}t}}{(E_{\gamma} - E_{\gamma'})^3 (E_{\gamma_1} - E_{\gamma'})} \right] |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'}. \quad (244)$$

$$A_6(ccnnn, kkc; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma}t} |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma'}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma'})}. \quad (245)$$

$$A_6(ccnnn, kkc; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma'}t} |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_1} \eta_{\gamma\gamma_2}}{(E_{\gamma} - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})}. \quad (246)$$

$$A_6(ccnnn, kkc; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma}t} |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma} \eta_{\gamma\gamma_2} \eta_{\gamma_1\gamma_3} \delta_{\gamma\gamma'}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma_2})(E_{\gamma} - E_{\gamma_3})}. \quad (247)$$

$$A_6(cncnn, kkkc, kck; t^2e, t^3e) = \sum_{\gamma_1} \frac{(-it)^2 e^{-iE_\gamma t} |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}}{2 (E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma'})^2}. \quad (248)$$

$$A_6(cncnn, kkkc, knk; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2 e^{-iE_\gamma t} |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma_1\gamma_2} \eta_{\gamma_1\gamma'}}{2 (E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma'})^2}. \quad (249)$$

$$A_6(cncnn, kkkn, kck; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2 e^{-iE_\gamma t} |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma_2}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'} \eta_{\gamma_1\gamma_2} \eta_{\gamma_2\gamma'}}{2 (E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma'})}. \quad (250)$$

$$A_6(cncnn, kkkn, knk, kc; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2 e^{-iE_\gamma t} |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma_1\gamma_2} \eta_{\gamma_1\gamma'}}{2 (E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma'})^2}. \quad (251)$$

$$\begin{aligned} & A_6(cncnn, kkkn, knk, kn; t^2e, t^3e) \\ &= \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{(-it)^2 e^{-iE_\gamma t} |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma_2}|^2 g_1^{\gamma\gamma_3} g_1^{\gamma_3\gamma'} \eta_{\gamma\gamma'} \eta_{\gamma_1\gamma_2} \eta_{\gamma_1\gamma_3} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma_3} \eta_{\gamma_2\gamma'}}{2 (E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma_3}) (E_\gamma - E_{\gamma'})}. \end{aligned} \quad (252)$$

$$A_6(cnnnc, kckk, kkc; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2 e^{-iE_\gamma t} |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma_2}|^2 |g_1^{\gamma\gamma_2}|^2 \delta_{\gamma\gamma'}}{2 (E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2})^2} \quad (253)$$

$$A_6(cnnnc, knkk, kkc; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{(-it)^2 e^{-iE_\gamma t} |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma_2}|^2 |g_1^{\gamma_2\gamma_3}|^2 \eta_{\gamma\gamma_3} \eta_{\gamma_1\gamma_2} \eta_{\gamma_1\gamma_3} \delta_{\gamma\gamma'}}{2 (E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2})^2 (E_\gamma - E_{\gamma_3})}. \quad (254)$$

$$\begin{aligned} A_6(cnnnc, kcck; t^2e, t^3e) &= \sum_{\gamma_1} \left[\frac{(-it)^2}{2} \frac{e^{-iE_\gamma t}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma'})^3} \right. \\ &\quad \left. + \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^3 (E_{\gamma_1} - E_{\gamma'})} \right] |g_1^{\gamma\gamma'}|^4 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}. \end{aligned} \quad (255)$$

$$A_6(cnnnc, kcnk; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2 e^{-iE_{\gamma'} t} |g_1^{\gamma\gamma'}|^2 |g_1^{\gamma_2\gamma'}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma_1\gamma_2}}{2 (E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})}. \quad (256)$$

$$A_6(cnnnc, knck; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2 e^{-iE_\gamma t} |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma_1\gamma_2} \eta_{\gamma_1\gamma'}}{2 (E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2}) (E_{\gamma_2} - E_{\gamma'})^2}. \quad (257)$$

$$A_6(nccnn, kkkc, ck; t^2e, t^3e) = \sum_{\gamma_1} \frac{(-it)^2 e^{-iE_{\gamma'} t} |g_1^{\gamma_1\gamma'}|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'}}{2 (E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})^2}. \quad (258)$$

$$A_6(nccnn, kkkc, nk; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2 e^{-iE_{\gamma'} t} |g_1^{\gamma_1\gamma'}|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_1} \eta_{\gamma\gamma_2}}{2 (E_\gamma - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'})^2 (E_{\gamma_2} - E_{\gamma'})}. \quad (259)$$

$$A_6(ncncn, ckc; t^2e, t^3e) = \sum_{\gamma_1} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1}t} |g_1^{\gamma_1}|^2 |g_1^{\gamma_1\gamma'}|^2 g_1^{\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma'})^2}. \quad (260)$$

$$A_6(ncncn, ckn; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1}t} |g_1^{\gamma_1}|^2 |g_1^{\gamma_1\gamma_2}|^2 g_1^{\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma'} \eta_{\gamma_2\gamma'}}{(E_\gamma - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2}) (E_{\gamma_1} - E_{\gamma'})}. \quad (261)$$

$$A_6(ncncn, nkc; t^2e, t^3e) = - \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1}t} |g_1^{\gamma_1\gamma'}|^2 |g_1^{\gamma_1\gamma_2}|^2 g_1^{\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma'} \eta_{\gamma_2\gamma'}}{(E_\gamma - E_{\gamma_1}) (E_{\gamma_1} - E_{\gamma_2}) (E_{\gamma_1} - E_{\gamma'})^2}. \quad (262)$$

$$A_6(ncncn, nkn, c; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1}t} |g_1^{\gamma_1}|^2 |g_1^{\gamma_1\gamma_2}|^2 |g_1^{\gamma_1\gamma_3}|^2 \eta_{\gamma\gamma_2} \eta_{\gamma\gamma_3} \eta_{\gamma_2\gamma_3} \delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2}) (E_{\gamma_1} - E_{\gamma_3})}. \quad (263)$$

$$\begin{aligned} & A_6(ncncn, nkn, n; t^2e, t^3e) \\ &= - \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1}t} |g_1^{\gamma_1\gamma_2}|^2 |g_1^{\gamma_1\gamma_3}|^2 g_1^{\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma_3} \eta_{\gamma\gamma'} \eta_{\gamma_2\gamma_3} \eta_{\gamma_2\gamma'} \eta_{\gamma_3\gamma'}}{(E_\gamma - E_{\gamma_1}) (E_{\gamma_1} - E_{\gamma_2}) (E_{\gamma_1} - E_{\gamma_3}) (E_{\gamma_1} - E_{\gamma'})}. \end{aligned} \quad (264)$$

$$A_6(ncnnc, kkck, ckk; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1}t} |g_1^{\gamma_1}|^2 |g_1^{\gamma_2}|^2 |g_1^{\gamma_1\gamma_2}|^2 \delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2})^2}. \quad (265)$$

$$A_6(ncnnc, kknk, ckk; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1}t} |g_1^{\gamma_1}|^2 |g_1^{\gamma_1\gamma_3}|^2 |g_1^{\gamma_1\gamma_2}|^2 \eta_{\gamma\gamma_2} \eta_{\gamma_1\gamma_3} \eta_{\gamma_2\gamma_3} \delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma_3})}. \quad (266)$$

$$A_6(nncn, cc; t^2e, t^3e) = \sum_{\gamma_1} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1}t} |g_1^{\gamma_1}|^2 g_1^{\gamma_1\gamma'} g_1^{\gamma_1\gamma_1} g_1^{\gamma_1\gamma'} g_1^{\gamma_1\gamma'}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma'})^2}. \quad (267)$$

$$A_6(nncn, cn; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1}t} |g_1^{\gamma_1\gamma_2}|^2 g_1^{\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma_1\gamma'} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma'}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2})^2 (E_\gamma - E_{\gamma'})}. \quad (268)$$

$$A_6(nncn, ckkk, kck; t^2e, t^3e) = \sum_{\gamma_1} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1}t} |g_1^{\gamma_1\gamma'}|^2 |g_1^{\gamma_1\gamma'}|^2 g_1^{\gamma_1} g_1^{\gamma_1\gamma'}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})^2}. \quad (269)$$

$$A_6(nncn, ckkk, knk; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1}t} |g_1^{\gamma_1\gamma'}|^2 |g_1^{\gamma_2\gamma'}|^2 g_1^{\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma_1\gamma_2}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})}. \quad (270)$$

$$A_6(nncn, nkkk, kck; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1}t} |g_1^{\gamma_1\gamma'}|^2 |g_1^{\gamma_2\gamma'}|^2 g_1^{\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma_1\gamma_2} \eta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'})^2 (E_{\gamma_2} - E_{\gamma'})}. \quad (271)$$

$$A_6(nncn, nkkk, knk, ck; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma_1}t} |g_1^{\gamma_1\gamma'}|^2 |g_1^{\gamma_2\gamma'}|^2 g_1^{\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma_1\gamma_2}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})}. \quad (272)$$

$$\begin{aligned}
& A_6(nncnc, nkkk, knk, nk; t^2e, t^3e) \\
&= \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma'}t} |g_1^{\gamma_2\gamma'}|^2 |g_1^{\gamma_3\gamma'}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma_3} \eta_{\gamma\gamma'} \eta_{\gamma_1\gamma_2} \eta_{\gamma_1\gamma_3} \eta_{\gamma_2\gamma_3}}{(E_{\gamma} - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'}) (E_{\gamma_3} - E_{\gamma'})}.
\end{aligned} \tag{273}$$

$$\begin{aligned}
A_6(nnncc, cckk; t^2e, t^3e) &= \sum_{\gamma_1} \left[\frac{(-it)^2}{2} \frac{e^{-iE_{\gamma}t}}{(E_{\gamma} - E_{\gamma_1}) (E_{\gamma} - E_{\gamma'})^3} \right. \\
&\quad \left. + \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma'}t}}{(E_{\gamma} - E_{\gamma'})^3 (E_{\gamma_1} - E_{\gamma'})} \right] |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'}.
\end{aligned} \tag{274}$$

$$A_6(nnncc, cnkk; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma}t} |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma'}}{(E_{\gamma} - E_{\gamma_1}) (E_{\gamma} - E_{\gamma_2}) (E_{\gamma} - E_{\gamma'})^2}. \tag{275}$$

$$A_6(nnncc, nckk; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma'}t} |g_1^{\gamma_2\gamma'}|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_1} \eta_{\gamma\gamma_2}}{(E_{\gamma} - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})^2}. \tag{276}$$

$$A_6(nnncc, nnkk, c; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma}t} |g_1^{\gamma\gamma_3}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma} \eta_{\gamma\gamma_2} \eta_{\gamma_1\gamma_3} \delta_{\gamma\gamma'}}{(E_{\gamma} - E_{\gamma_1}) (E_{\gamma} - E_{\gamma_2}) (E_{\gamma} - E_{\gamma_3})^2}. \tag{277}$$

$$A_6(cnnnn, kccc; t^2e, t^3e) = \sum_{\gamma_1} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma}t} |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma'})^2}. \tag{278}$$

$$A_6(cnnnn, kccn; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma}t} |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma'}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma_2}) (E_{\gamma} - E_{\gamma'})}. \tag{279}$$

$$A_6(cnnnn, kncc; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma}t} |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma}}{(E_{\gamma} - E_{\gamma_1}) (E_{\gamma} - E_{\gamma_2}) (E_{\gamma} - E_{\gamma'})^2}. \tag{280}$$

$$A_6(cnnnn, kcnn, kkc; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma}t} |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma_1} g_1^{\gamma_1\gamma_3} g_1^{\gamma_3\gamma} \eta_{\gamma_2\gamma_3} \delta_{\gamma\gamma'}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma_2}) (E_{\gamma} - E_{\gamma_3})}. \tag{281}$$

$$A_6(cnnnn, kncn, kc; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma}t} |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma'}}{(E_{\gamma} - E_{\gamma_1}) (E_{\gamma} - E_{\gamma_2}) (E_{\gamma} - E_{\gamma'})^2}. \tag{282}$$

$$\begin{aligned}
& A_6(cnnnn, kncn, kn; t^2e, t^3e) \\
&= \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma}t} |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma} g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma_2} \eta_{\gamma_1\gamma_3} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma'} \eta_{\gamma_3\gamma'}}{(E_{\gamma} - E_{\gamma_1}) (E_{\gamma} - E_{\gamma_2}) (E_{\gamma} - E_{\gamma_3}) (E_{\gamma} - E_{\gamma'})}.
\end{aligned} \tag{283}$$

$$A_6(cnnnn, knnn, kcc; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma}t} |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma_1} g_1^{\gamma_1\gamma} \eta_{\gamma\gamma_3} \eta_{\gamma_1\gamma_2} \delta_{\gamma\gamma'}}{(E_{\gamma} - E_{\gamma_1})^2 (E_{\gamma} - E_{\gamma_2}) (E_{\gamma} - E_{\gamma_3})}. \tag{284}$$

$$A_6(cnnnn, knnn, kn; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2, \gamma_3, \gamma_4} \frac{(-it)^2 e^{-iE_\gamma t} |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma_4} g_1^{\gamma_4\gamma} \eta_{\gamma\gamma_3} \eta_{\gamma_1\gamma_2} \eta_{\gamma_1\gamma_3} \eta_{\gamma_1\gamma_4} \eta_{\gamma_2\gamma_4} \delta_{\gamma\gamma'}}{2 (E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma_3}) (E_\gamma - E_{\gamma_4})}. \quad (285)$$

$$A_6(ncnnn, kkcc, ckk; t^2e, t^3e) = \sum_{\gamma_1} \frac{(-it)^2 e^{-iE_{\gamma'} t} |g_1^{\gamma\gamma'}|^2 |g_1^{\gamma_1\gamma'}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}}{2 (E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})^2}. \quad (286)$$

$$A_6(ncnnn, kkcc, nkk; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2 e^{-iE_{\gamma'} t} |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_2} g_1^{\gamma_2\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma_1} \eta_{\gamma_1\gamma_2}}{2 (E_\gamma - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'})^2 (E_{\gamma_2} - E_{\gamma'})}. \quad (287)$$

$$A_6(ncnnn, kknc, ckk; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2 e^{-iE_{\gamma'} t} |g_1^{\gamma\gamma'}|^2 |g_1^{\gamma_1\gamma'}|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_1} \eta_{\gamma_1\gamma_2}}{2 (E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})}. \quad (288)$$

$$A_6(ncnnn, kknc, nkk, ck; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2 e^{-iE_{\gamma'} t} |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma_1} \eta_{\gamma_1\gamma_2}}{2 (E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})}. \quad (289)$$

$$A_6(ncnnn, kknc, nkk, nk; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{(-it)^2 e^{-iE_{\gamma'} t} |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma'} \eta_{\gamma\gamma_1} \eta_{\gamma_1\gamma_2} \eta_{\gamma_1\gamma_3} \eta_{\gamma_1\gamma_2} \eta_{\gamma_1\gamma_3}}{2 (E_\gamma - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'}) (E_{\gamma_3} - E_{\gamma'})}. \quad (290)$$

$$A_6(nncnn, ckcc, kck; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2 e^{-iE_\gamma t} |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma_2}|^2 |g_1^{\gamma_1\gamma_2}|^2 \delta_{\gamma\gamma'}}{2 (E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2})^2}. \quad (291)$$

$$A_6(nncnn, ckcc, knk; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{(-it)^2 e^{-iE_\gamma t} |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma} \eta_{\gamma_1\gamma_3} \delta_{\gamma\gamma'}}{2 (E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2})^2 (E_\gamma - E_{\gamma_3})}. \quad (292)$$

$$A_6(nnnncn, cckk, kkc; t^2e, t^3e) = \sum_{\gamma_1} \frac{(-it)^2 e^{-iE_\gamma t} |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}}{2 (E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma'})^2}. \quad (293)$$

$$A_6(nnnncn, cckk, kn; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2 e^{-iE_\gamma t} |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma_1\gamma'} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma'}}{2 (E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma'})}. \quad (294)$$

$$A_6(nnnncn, cnkk, kkc; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2 e^{-iE_\gamma t} |g_1^{\gamma\gamma_2}|^2 |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma_1\gamma_2} \eta_{\gamma_2\gamma'}}{2 (E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma'})^2}. \quad (295)$$

$$A_6(nnnncn, cnkk, kn, kc; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2 e^{-iE_\gamma t} |g_1^{\gamma\gamma_2}|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma_1\gamma'} \eta_{\gamma_1\gamma_2} \eta_{\gamma_2\gamma'}}{2 (E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma'})^2}. \quad (296)$$

$$\begin{aligned}
& A_6(nnnncn, cnkk, kkn, kn; t^2e, t^3e) \\
&= \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t} |g_1^{\gamma\gamma_3}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma_3} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma_3} \eta_{\gamma_2\gamma'} \eta_{\gamma_3\gamma'}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})(E_\gamma - E_{\gamma'})}. \tag{297}
\end{aligned}$$

$$A_6(nnnnc, ccck; t^2e, t^3e) = \sum_{\gamma_1} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma'} t} |g_1^{\gamma_1\gamma'}|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})^2}. \tag{298}$$

$$A_6(nnnnc, ccnk; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma'} t} |g_1^{\gamma_2\gamma'}|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'} \eta_{\gamma_2\gamma} \eta_{\gamma_1\gamma_2}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})}. \tag{299}$$

$$A_6(nnnnc, ncck; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma'} t} |g_1^{\gamma_1\gamma'}|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma_1\gamma} \eta_{\gamma_2\gamma}}{(E_\gamma - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'})^2 (E_{\gamma_2} - E_{\gamma'})}. \tag{300}$$

$$A_6(nnnnc, ncnk, ck; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma'} t} |g_1^{\gamma\gamma'}|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma_1\gamma} \eta_{\gamma_2\gamma}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'})}. \tag{301}$$

$$\begin{aligned}
& A_6(nnnnc, ncnk, nk; t^2e, t^3e) \\
&= \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{(-it)^2}{2} \frac{e^{-iE_{\gamma'} t} |g_1^{\gamma_3\gamma'}|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma_1\gamma} \eta_{\gamma_2\gamma} \eta_{\gamma_3\gamma} \eta_{\gamma_1\gamma_3} \eta_{\gamma_2\gamma_3}}{(E_\gamma - E_{\gamma'}) (E_{\gamma_1} - E_{\gamma'}) (E_{\gamma_2} - E_{\gamma'}) (E_{\gamma_3} - E_{\gamma'})}. \tag{302}
\end{aligned}$$

$$A_6(nnnnc, nnck, c; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t} |g_1^{\gamma\gamma_2}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma} \eta_{\gamma_1\gamma_3} \delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})^2 (E_\gamma - E_{\gamma_3})}. \tag{303}$$

$$A_6(nnnnc, nnnk, cck; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t} |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma} \eta_{\gamma_2\gamma} \eta_{\gamma_1\gamma_3} \delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma_3})}. \tag{304}$$

$$\begin{aligned}
& A_6(nnnnc, nnnk, cnk; t^2e, t^3e) \\
&= \sum_{\gamma_1, \gamma_2, \gamma_3, \gamma_4} \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t} |g_1^{\gamma\gamma_4}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma} \eta_{\gamma_2\gamma} \eta_{\gamma_3\gamma} \eta_{\gamma_1\gamma_3} \eta_{\gamma_1\gamma_4} \eta_{\gamma_2\gamma_4} \eta_{\gamma_3\gamma_4} \delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})(E_\gamma - E_{\gamma_4})}. \tag{305}
\end{aligned}$$

$$A_6(nnnnn, cccc; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2} \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t} (g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma})^2 \delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2})^2}. \tag{306}$$

$$A_6(nnnnn, ccnc; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_3} g_1^{\gamma_3\gamma} \eta_{\gamma_2\gamma_3} \delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma_3})}. \tag{307}$$

$$A_6(nnnnn, cncc; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma_3} g_1^{\gamma_3\gamma_2} g_1^{\gamma_2\gamma} \eta_{\gamma_1\gamma_3} \delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2})^2 (E_\gamma - E_{\gamma_3})}. \tag{308}$$

$$A_6(nnnnn, cnnc, kck; t^2e, t^3e) = \sum_{\gamma_1, \gamma_2, \gamma_3} \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma_3} g_1^{\gamma_3\gamma_1} g_1^{\gamma_1\gamma} \eta_{\gamma_2\gamma_3} \delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma_3})}. \tag{309}$$

$$\begin{aligned}
& A_6(nnnnn, cnnc, knk; t^2e, t^3e) \\
&= \sum_{\gamma_1, \gamma_2, \gamma_3, \gamma_4} \frac{(-it)^2}{2} \frac{e^{-iE_\gamma t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma_3} g_1^{\gamma_3\gamma_4} g_1^{\gamma_4\gamma} \eta_{\gamma_1\gamma_3} \eta_{\gamma_1\gamma_4} \eta_{\gamma_2\gamma_3} \eta_{\gamma_2\gamma_4} \delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1}) (E_\gamma - E_{\gamma_2}) (E_\gamma - E_{\gamma_3}) (E_\gamma - E_{\gamma_4})}. \tag{310}
\end{aligned}$$